

# Splitting of separatrices in a family of area-preserving maps that unfolds a fixed point at the resonance of order three

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## Abstract

We study the exponentially small splitting of separatrices in an one parameter family of area-preserving maps that unfolds the 1:3 resonance. We show that under a certain non-degeneracy condition we can compute a Stokes constant  $\theta$  for the map. When this constant is non zero, we provide an asymptotic for the splitting of separatrices for the map.

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# 1 Introduction

In this paper we will look at the splitting of separatrices in area-preserving maps close to 1:3 resonance. The main motivation for the study of area-preserving maps on the plane comes for the question of stability of periodic orbits in Hamiltonian systems with 2 degrees of freedom. The state space of such a system is of dimension 4, since the Hamiltonian is an integral of the system, by choosing a value for this function we can look at the surface that this defines and this drops the dimension to 3. Then we assume there exists a periodic orbit on this 3 dimensional surface and we construct the first return map of this orbit. This map is 2-dimensional and we typically move the fixed point that corresponds to the periodic orbit to the origin. This map, defined on a neighbourhood of the origin on the plane, can be shown to preserve area, see [Arn78].

If the multipliers of the fixed point are on the unit circle, the point is called elliptic. In a neighbourhood of such fixed point any such map can be approximated by a Hamiltonian flow. This implies that for every map we can construct a formal integral of the original Hamiltonian system. So, if the reduction to the normal form is analytic, then the Hamiltonian system is integrable. In the book [AKN06] the normal form of all elliptic cases can be found.

Arnold, in the appendix of his book *Mathematical Methods of Classical Mechanics*, [Arn78], conjectures that Hamiltonian systems with resonant periodic orbits are in general non-integrable. He uses a Hamiltonian system with a periodic orbit at 1:3 resonance as an example. Arnold conjectured that in general the separatrices of the hyperbolic points do not coincide but intersect transversely. He also states that the difference of the separatrices has to be exponential small, since if this was not the case, the normal form would be able to detect it.

Let  $U \subset \mathbb{C}^2$  be a neighbourhood of the origin in  $\mathbb{C}^2$ . Then we call an *area-*

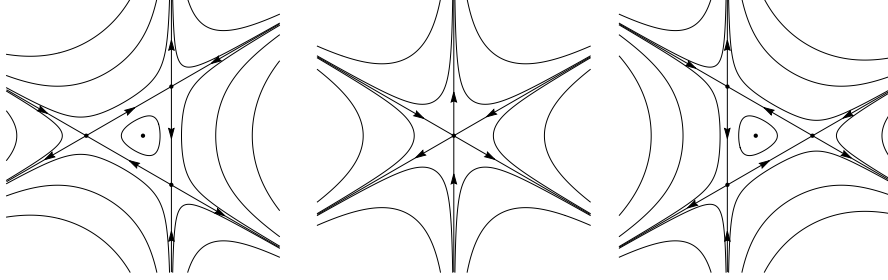


Figure 1: The normal form of the unfolded 1:3 resonant map.

preserving map on  $U$  any  $g : U \rightarrow \mathbb{C}^2$  such that  $g(0) = 0$  and  $\det g'(z) = 1$  for all  $z \in U$ . Moreover, if  $g$  is analytic in  $U$  and  $g(U \cap \mathbb{R}^2) \subset \mathbb{R}^2$  we call  $g$  a *real analytic area-preserving map on  $U$* . Usually the existence of the set  $U$  will be implicitly assumed.

Let  $U \subset \mathbb{C}^2$  be a neighbourhood of the origin in  $\mathbb{C}^2$  and  $V$  a neighbourhood of the origin in  $\mathbb{C}$ . Let  $g_\mu$  be an analytic area-preserving map on  $U$  for all  $\mu \in V$ , such that  $g_\mu$  is real analytic if  $\mu$  is real. Then if  $g_\mu$  depends analytically on  $\mu$  we call  $g_\mu$  an *analytic family of real analytic area-preserving maps on  $U$* .

Let  $g$  be an area-preserving map, if  $g'(0)$  is the identity then we call  $g$  an *tangent to identity map*.

Let  $g_0$  be a real analytic area-preserving map such that  $g'_0(0)$  has eigenvalues  $\epsilon^\pm = e^{\pm 2\pi i/3}$ . We will call such  $g_0$  a *real analytic area-preserving map at 1:3 resonance*.

**Theorem 1.1** (Birkhoff normal form). *There is a formal canonical change of coordinates  $\Phi$  such that the map  $N = \Phi \circ g_0 \circ \Phi^{-1}$  commutes with the rotation  $R_{2\pi/3}$ , i.e.:  $N \circ R_{2\pi/3} = R_{2\pi/3} \circ N$ .*

The map  $N$  is called a *Birkhoff normal form* of  $g_0$  (see [Bir66]). The map  $R_{4\pi/3} \circ N$  is tangent to identity. A tangent to the identity symplectic map can be formally represented as a time-one map of an autonomous Hamiltonian system: there is a formal Hamiltonian  $H$  such that

$$N = R_{2\pi/3} \circ \phi_H^1$$

where  $\phi_H^1$  is the formal time one flow of the Hamiltonian vector field that corresponds to the Hamiltonian  $H$ . The Hamiltonian vector field is usually called *Takens normal form*. This implies that  $H$  is a formal integral of the map  $N$ . Reverting back to the original coordinates we get a formal integral of the map  $g_0$ . This implies that if  $g_0$  is not integrable then at least one of the formal series  $\Phi$  and  $H$  has to be divergent.

We say that a real analytic area-preserving map at 1:3 resonance is *non-degenerate* if the terms of order 3 in the Hamiltonian  $H$  do not vanish. Vanishing of the third order terms is a codimension 2 condition, see [GG09].

The normal form of such family of maps can be seen in Figure 1. We see that close to the resonance there exists a 3-periodic orbit with homoclinic connec-

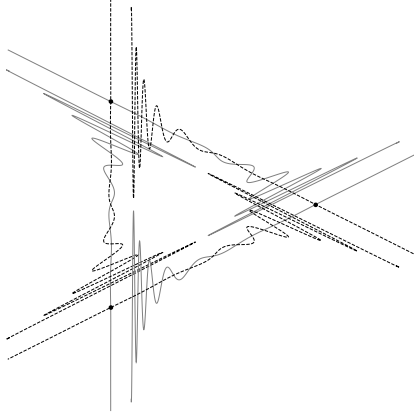


Figure 2: The splitting of the separatrices of a map close to 1:3 resonance.

tions. The separatrices of this orbit define an invariant triangle. The main result of the present paper is that under some mild conditions the homoclinic connections of the normal form do not exist in the dynamics of the map and the separatrices intersect in a non-trivial way. In the present paper we prove that under the assumption that the map is non-degenerate and that the Stokes constant of the resonant map does not vanish, the separatrices split and their difference is actually exponentially small. The configuration of the separatrices can be seen in figure 2.

In order to measure the splitting of the separatrices we will use *Lazutkin's homoclinic invariant*, introduced in [GLS94]. Notice that the points of intersection of the two separatrices form a homoclinic orbit. We define the homoclinic invariant to be the area of the parallelogram that is formed by the tangent vectors of the separatrices at a homoclinic point.

Let  $U \subset \mathbb{C}^2$  be a neighbourhood of the origin in  $\mathbb{C}^2$  and let  $g_\mu$  be an analytic family of real analytic area-preserving maps on  $U$  such that  $g_0$  is a non-degenerate area-preserving map at resonance 1:3. Let  $\lambda_\mu$  be an eigenvalue of the saddle points, if  $\frac{\partial}{\partial \mu} \lambda_\mu|_{\mu=0} \neq 0$  we will say that the family  $g_\mu$  *unfolds the resonance 1:3 generically*.

We define  $G_\mu := g_\mu^3$  and let  $V_0^+(t)$  and  $V_0^-(t)$  be parametrizations of a pair of stable and unstable invariant lines of the parabolic fixed point of the map  $G_0$  admitting the same asymptotic expansion, such that it holds  $V_0^\pm(t+1) = G_0(V_0^\pm(t))$ . We will call these two separatrices a *compatible pair of separatrices of the map  $G_0$* . The main result of the paper is the following theorem.

**Theorem 1.2.** *Let  $g_\mu$  be a real analytic, non-degenerate family of real analytic area-preserving maps that unfolds the resonance 1:3 generically. Let  $G_\mu = g_\mu^3$  be the third iterate of the map  $g_\mu$  and let  $V_0^\pm(t)$  be a compatible pair of separatrices of the map  $G_0$ .*

Then the limit

$$\theta = \lim_{\operatorname{Im} t \rightarrow -\infty} e^{2\pi i t} \omega(V_0^+(t) - V_0^-(t), \dot{V}_0^-(t))$$

exists and it is called the Stokes constant of the map  $G_0$ .

For  $\mu \neq 0$  let  $\lambda_\mu$  denote the largest eigenvalue of its saddle points and let  $\Omega$  denote the Lazutkin homoclinic invariant of the map. If  $\theta$  does not vanish, then there exist  $\mu_0 > 0$  and real constants  $\vartheta_n$  such that for any  $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$  and any  $M \in \mathbb{N}$  it holds

$$\Omega(\mu) = \left( \sum_{n=0}^M \vartheta_n (\log \lambda_\mu)^n + O((\log \lambda_\mu)^{M+1}) \right) e^{-\frac{2\pi^2}{\log \lambda_\mu}}.$$

Moreover  $\vartheta_0 = 4\pi|\theta|$ .

A non-vanishing homoclinic invariant implies that the tangent vectors on the separatrices at the homoclinic point are not parallel, which of course it means that the intersection is transversal. This has the usual implication of the existence of a horseshoe, which in turn implies positive topological entropy. The theorem can also be applied to 1 degree of freedom time periodic Hamiltonian systems.

The homoclinic invariant was used to measure the splitting instead of other more intuitive quantities like the splitting angle or the splitting amplitude, because the latter depend on the coordinate system and on the homoclinic point chosen. On the other hand the area of the crescent, which is also used, is invariant under symplectic transformations and independent of the homoclinic point, however non-vanishing crescent area does not imply that the intersection is transversal.

These quantities can be obtained easily by the homoclinic invariant. The area of the crescent is a multiple of the homoclinic invariant, since the former is the integral and the later the derivative of the same almost<sup>1</sup> periodic function. Similarly the splitting amplitude is a multiple of the homoclinic invariant since the former is the supremum of the same function. The splitting angle can be computed by the homoclinic invariant using the length of the vector and the formula for the area of the parallelogram.

## 1.1 Historical remarks

The phenomenon we study here was first observed by the French mathematician Henri Poincaré around 1890 when investigating the stability of the solar system. Poincaré considered the system formed by three bodies: Sun, Earth and Moon, under the action of Newton's laws of gravity. In an attempt to prove the stability of the three body system, he used perturbation series and realized its divergent character [Poi90]. He also noticed that a small differences in the initial positions or velocities of one of the bodies would lead to a radically different state when compared to the unperturbed system, what is now commonly known

<sup>1</sup> We refer to the function  $\Theta^-$  defined in section 9. As we will see this function is approximated by a sinusoid function with exponentially small error.

as deterministic chaos. Poincaré realized that a small perturbation can destroy a homoclinic connection and its place is taken by a region where the stable and the unstable manifolds intersect in a highly non-trivial way. He was even able to prove for a concrete example that the width of this region was exponentially small with respect to the size of the perturbation.

This splitting of separatrices is exactly the phenomenon we are interested here. We will give some brief historical remarks and we encourage the reader to see the survey by Gelfreich and Lazutkin [GL01] for a more detailed exposition of the theory until 2000.

### 1.1.1 Splitting of separatrices in area-preserving maps

The obvious way to address the above question of stability of periodic orbits in two degrees of freedom systems is to look directly at the first return map.

The first map to be treated was the Chirikov standard map, defined on the torus by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \varepsilon \sin(x) \\ y + \varepsilon \sin(x) \end{pmatrix}.$$

For  $\varepsilon = 0$  the standard map is integrable but for  $\varepsilon > 0$  the homoclinic separatrix splits. An asymptotic formula for the splitting in the standard map was published by Lazutkin in 1984 in a pioneering article, see [Laz03] for the english translation. However the proof was incomplete and it was completed and published by Gelfreich in [Gel99]. The same asymptotic formula was derived by Hakim and Mallick, [HM93], using Ecalle's theory of resurgent functions. However their work was based on formal arguments.

Neishtadt, [Nei84], was the first to prove that the splitting in the difference of the two separatrices of analytic maps close to identity admits an exponentially small upper bound. Later Fontich and Simó, [FS90], using Lazutkin's methods gave a sharp upper bound.

Using the theory of resurgent functions, Gelfreich and Sauzin proved for an instance of the Hénon map at 1:1 resonance that the splitting of separatrices is exponentially small and provided the first asymptotic term for it, see [GS01].

More recently Martín, Sauzin and Seara have studied the splitting of separatrices in perturbations of the McMillan map, see [MSS11a] and [MSS11b]. Their approach combined the theory of resurgent functions with Lazutkin's original ideas.

A paper, [Gel02], was published by Gelfreich stating the first asymptotic term for the resonances 1:1, 1:2 and 1:3. However the only proof on these results published until now is a preprint by Brännström and Gelfreich [BG08]. There the authors derive and prove the asymptotic formula for area-preserving maps near a Hamiltonian saddle-centre bifurcation.

### 1.1.2 Non-autonomous perturbation of flows

An other way to address the above question of stability is to embed the first return map into the flow of an non-autonomous Hamiltonian system of one degree of freedom. This enables the usage of methods developed for differential equations and more results are available. This flow can be written as a periodic time dependent perturbation of an one degree of freedom Hamiltonian system. More precisely, we describe this system with the help of the Hamiltonian function

$$H(\mu, \varepsilon; x, y, t) = H_0(x, y) + \mu H_1(x, y, \frac{t}{\varepsilon}, \varepsilon, \mu),$$

with  $H_1(x, y, t, \varepsilon, \mu)$  periodic in  $t$ .

The natural question in this setting is whether homoclinic or heteroclinic connections that exists in the unperturbed system persist in the perturbed one.

The case where only  $\mu$  is considered to be a small parameter was solved using the so called Melnikov method, see [Mel63]. In this case we can reparametrize time such that  $\varepsilon = 1$ . Then for the separatrices of the system it holds

$$W_{\mu, \varepsilon}^{\pm}(t_0, t) = W_0(t - t_0) + \mu W_1^{\pm}(t_0, t) + O(\mu^2), \quad \pm t \in [t_0, \infty).$$

We define the *Melnikov function* by

$$M(t_0) := \int_{-\infty}^{\infty} \{H_0, H_1\}|_{W_0(t-t_0), t} dt,$$

where  $\{H, G\}$  is the Poisson bracket of  $H$  and  $G$ . For the difference between the two separatrices at  $t_0$ , measured in a coordinate system that uses  $H_0$  as the first of its coordinates, we get that the difference in the first component is

$$d(t_0) = \mu M(t_0) + O(\mu^2).$$

However, when both  $\mu$  and  $\varepsilon$  are considered small, then  $\varepsilon$  cannot be ignored. These systems are called *rapidly forced* systems since the period of the perturbation becomes arbitrarily small.

Nekhoroshev, [Nek77], showed that in many degrees of freedom Hamiltonian systems, the phase space can be covered by domains where the system behaves as if it was integrable for some time. He showed that this time is exponentially large with the size of the perturbation. Neishtadt showed in [Nei84] that  $d$  actually admits an upper bound that is exponentially small with  $\varepsilon$ . Neishtadt's results were refined by Treshchev in [Tre97]. Fontich based on Lazutkin's ideas, [Fon95], showed that the exponent depends on the location of the singularities in the parameter of the unperturbed separatrix.

In rapidly forced systems the Melnikov function can become exponentially small with  $\varepsilon$ , but since the error term is polynomially small in  $\mu$ , the error can become bigger than the approximation. This situation can be avoided of course when  $\mu$  is a function of  $\varepsilon$  which decreases exponentially as  $\varepsilon$  goes to 0. Then the error is also exponentially small and the Melnikov method can still be applied. It was shown in [Gel97b] and [DS97] that in systems where  $H_0(x, y) = y^2/2 + V'(x)$  and  $V''(0) \neq 0$  this can be relaxed to a polynomial dependence,  $|\mu| \leq C\varepsilon^p$ ,

with  $p$  big enough. A similar was proved in [BF04] for such systems but with  $V''(0) = 0$ .

Stronger results have been proved in specific systems. Poincaré [Poi93] discovered the phenomenon of splitting by looking at the system described by the Hamiltonian

$$\frac{y^2}{2} + \cos x + a \sin x \cos \frac{t}{\varepsilon}.$$

He proved that in this system the splitting is exponentially small and he derived the first term of the asymptotic expansion. Poincaré's arguments require  $a$  to be exponentially small in  $\varepsilon$  and his result is the same that the Melnikov method provides. However, for an  $\varepsilon$ -independent  $a$  Melnikov's method provides a wrong estimate. Treshchev [Tre96] and Gelfreich [Gel97a] independently showed that by obtaining a different asymptotic formula using the averaging method with a continuous parameter.

The most studied system has been the rapidly perturbed pendulum with a perturbation only depending on time,

$$\ddot{x} = \sin x + \mu \varepsilon^\eta \sin \frac{t}{\varepsilon}.$$

Many authors have published on this, gradually strengthening the result, see [HMS88], [Sch89], [DS92], [Ang93], [EKS93], [Gel94] and [Swa96].

Recently Gaivão and Gelfreich [GG11] used the generalized Swift-Hohenberg equation as an example to show the transversality of the homoclinic solutions near a Hamiltonian-Hopf bifurcation.

Baldoma, Fontich, Guardia and Seara [BFGS12] showed that in systems where  $H_0 = \frac{y^2}{2} + V(x)$  with  $V$  an algebraic or trigonometric polynomial and  $|\mu| \leq C\varepsilon^\eta$ , the Melnikov method can be applied if  $\eta > 0$ . Moreover, they also showed that the Melnikov method fails when  $p$  becomes zero and they derived the first term of the asymptotic series in this case. Guardia [Gua13] showed that this result holds also close to the resonances of this Hamiltonian.

### 1.1.3 Splitting of separatrices in physics

The same phenomenon has been studied in physics although in a different framework. The common technique there is truncating an asymptotic series in the optimal order and then showing that the remainder is exponentially small. This technique is called asymptotics beyond all order or superasymptotics, see [Ber91], [STL12] or [IL05].

There exist many examples of problems for which asymptotic power series methods lead to divergent series. Oppenheimer [Opp28] while investigating a phenomenon in quantum physics known as the Stark effect, demonstrated that the lifetime of a certain quantum state was inversely proportional to a quantity exponentially small with the strength of the electric field applied at the system.

Kruskall and Segur [KS91] demonstrated that the geometric model for dendritic crystal growth fails to produce needle crystal solutions due to exponentially small effects, a byproduct of the breakage of a heteroclinic connection. This work



has influenced many others in the field and the same technique has been applied at the formal level to prove the non-persistence of homoclinic or heteroclinic solutions to certain singularly perturbed systems. Examples of application of this method include surface tension and wave formation [GJ95], [YA97], [Tov00], [VdBK09], [TCVB11]), crystal growth [CM05] and optics [CK09]. More information about applications of exponentially small splitting to mechanics, fluids and optics can be found in the survey of Champneys [Cha98].

In his book [Lom00], Lombardi puts the superasymptotics into rigorous arguments that can be used to solve many problems in exponentially small phenomena. He did that by reducing the problem to the study of certain oscillatory integrals which describe the exponentially small terms. He applied his method to water waves.

## 2 Unique normal forms for families of maps

The formal Hamiltonian  $H$  constructed for the normal form is not defined uniquely so there is room for further normalization.

**Proposition 2.1** ([GG09]). *Let  $f_0$  be a non-degenerate, area-preserving map at 1:3 resonance. Then there is a formal Hamiltonian  $H$  and formal canonical change of variables which conjugates  $f_0$  with  $R_{2\pi/3} \circ \phi_H^1$ . Moreover,  $H$  has the following form:*

$$H(x, y) = (x^2 + y^2)^3 A(x^2 + y^2) + (2x^3 - 6xy^2) B(x^2 + y^2), \quad (1)$$

where  $A$  and  $B$  are series in one variable with real coefficients:

$$A(I) = \sum_{\substack{k \geq 0 \\ k \not\equiv 2 \pmod{3}}} a_k I^k, \quad B(I) = \frac{b_0}{6} + \sum_{\substack{k \geq 1 \\ k \not\equiv 2 \pmod{3}}} b_k I^k$$

and the coefficient of  $A$  and  $B$  are uniquely defined if  $b_0 \neq 0$

For the coefficient  $b_0$  it holds  $b_0 = 6|h_{30}|$ , where  $h_{30}$  is the 3rd order coefficient in the Takens normal form Hamiltonian.

For a real analytic family of maps,  $f_\mu$ , such that  $f_0$  is a map at 1:3 resonance we have the following result.

**Proposition 2.2** ([GG09]). *Let  $f_\mu$  be an analytic family of area-preserving maps such that  $f_0$  is at resonance 1:3 and let the coefficient  $b_0$  for the map  $f_0$  not vanish. Then there is a formal Hamiltonian  $H$  and formal canonical change of variables which conjugates  $f_\mu$  with  $R_{2\pi/3} \circ \phi_H^1$ . Moreover,  $H$  has the following form:*

$$H(\mu; x, y) = (x^2 + y^2) A(\mu, x^2 + y^2) + (2x^3 - 6xy^2) B(\mu, x^2 + y^2),$$

where  $A$  and  $B$  are series in two variables with real coefficients:

$$A(\mu, I) = \sum_{\substack{k, m \geq 0 \\ k \not\equiv 1 \pmod{3}}} a_{k, m} I^k \mu^m, \quad B(\mu, I) = \frac{b_{0,0}}{6} + \sum_{\substack{k, m \geq 1 \\ k \not\equiv 2 \pmod{3}}} b_{k, m} I^k \mu^m,$$

with  $b_{0,0} = b_0$  and  $a_{0,0} = a_{1,0} = 0$ . Moreover the coefficients of these series are unique.

**Remark.** Since the formal Hamiltonian  $H_\mu$  is invariant under the rotation  $R_{2\pi/3}$  we have that  $N^3 = (R_{2\pi/3} \circ \phi_H^1)^3 = \phi_H^3 = \phi_{3H}^1$ . Then from the relation  $N_\mu^3 = \Phi_\mu \circ f_\mu^3 \circ \Phi_\mu^{-1}$  we see that the third iterate of the map  $f_\mu$  can be described by the same normal form as  $f_\mu$ .

### 3 Setup

Let  $U \subset \mathbb{C}^2$  be a neighbourhood of the origin in  $\mathbb{C}^2$  and let  $f_\mu$  be an analytic family of real analytic area-preserving maps that unfolds the resonance 1:3 generically. We fix  $M \in \mathbb{N}$  and define  $F_\mu = f_\mu^3$ . We assume that  $F_\mu$  agrees with the normal form in Proposition 2.2 up to degree  $N = 6M + 39$ . Given such family we can apply the following theorem to the map  $f_0$ .

**Theorem 3.1.** *Let  $f_0$  be a non-degenerate area-preserving map at 1:3 resonance that agrees with the normal form at least up to order 4. We define  $F_0 = f_0^3$ . Then there exists a unique formal solution with real coefficients,*

$$\tilde{W}_0(t) = \begin{pmatrix} 0 \\ -\frac{1}{b_0 t} \end{pmatrix} + O(|t|^{-3}) \in \frac{1}{t} \mathbb{C} \left[ \left[ \frac{1}{t} \right] \right]^2,$$

of the equation

$$\tilde{W}_0(t+1) = F_0(\tilde{W}_0(t)) \quad (2)$$

and any other formal solution of the form  $(0, -\frac{1}{b_0 t}) + O(|t|^{-2})$  can be written as  $\tilde{W}_0(t+c)$  for some  $c \in \mathbb{C}$ . Moreover there exists a formal solution with real coefficients,  $\tilde{\Xi}_0 \in t^2 \mathbb{C} \left[ \left[ \frac{1}{t} \right] \right]^2$ , of the equation

$$X(t+1) = F'_0(\tilde{W}_0(t)) \cdot X(t),$$

such that

$$\tilde{\Xi}_0(t) = \begin{pmatrix} b_0 t^2 - 18 \frac{b_1}{b_0^2} + \frac{24b_1^2}{b_0^3} t^{-2} \\ -\frac{8a_0}{b_0^3} t^{-1} \end{pmatrix} + O(|t|^{-3}),$$

and  $\det(\tilde{\Xi}_0(t), \dot{\tilde{W}}_0(t)) = 1$ .

There are two solutions of the equation (2),  $W_0^+$  and  $W_0^-$ , that admit  $\tilde{W}_0$  as asymptotic and satisfy  $\lim_{t \rightarrow \pm\infty} W_0^\pm(t) = 0$  and there exist two complex constants,  $\theta$  and  $\rho$ , such that for any  $t \in \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 1, \operatorname{Im}(z) < 0\}$ , with  $|t|$  big enough, it holds

$$W_0^+(t) - W_0^-(t) \asymp e^{-2\pi i t} \left( \theta \tilde{\Xi}(t) + \rho \dot{\tilde{W}}(t) \right) + O(t^7 e^{-4\pi i t}) \quad (3)$$

and

$$\theta = \lim_{t \rightarrow +\infty} e^{2\pi i t} \omega(W_0^+(-it) - W_0^-(-it), \dot{W}_0^-(-it)).$$

A proof of this theorem can be found in [GM17]. Even though the constants  $\theta$  and  $\rho$  have the same role, we will focus on  $\theta$  and we will call it *the Stokes constant of  $F_0$* .

Choosing a small  $\mu \neq 0$  we get three saddle points with  $\lambda_\mu > 1$  being their largest eigenvalue. We set  $\varepsilon = \log(\lambda_\mu)$  and by the implicit function theorem we can write the parameter  $\mu$  as a function of  $\varepsilon$ . Throughout the text we assume that the parametrization of  $F_\mu$  is changed from  $\mu$  to  $\varepsilon$  which is in a sense the natural parametrization. Notice that by definition  $\varepsilon$  is always positive, however this does not restrict the generality of the result as we can consider the cases  $\mu \in (0, \mu_0)$  and  $\mu \in (-\mu_0, 0)$  separately.

**Theorem 3.2.** *Let  $F_\varepsilon$  be an analytic family of area-preserving maps as described above and let  $\Omega$  be the Lazutkin homoclinic invariant of the map. If the Stokes constant  $\theta$  of the map  $F_0$  does not vanish, then there exist  $\varepsilon_0 > 0$  and real constants  $\vartheta_n$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  it holds*

$$\Omega(\varepsilon) = \left( \sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1}) \right) e^{-\frac{2\pi^2}{\varepsilon}}.$$

Moreover  $\vartheta_0 = 4\pi|\theta|$ .

*proof of Theorem 1.2.* We consider a family of area-preserving maps,  $g_\mu$ , as described in the Theorem 1.2 and we set  $G_\mu = g_\mu^3$  and we fix  $M \in \mathbb{N}$ . Let  $\Phi$  be a canonical transformation that brings  $G_\mu$  in agreement with the normal form up to order  $6M + 39$ . Since the order is finite the transformation  $\Phi$  is analytic. Let  $F_\mu = \Phi^{-1} \circ G_\mu \circ \Phi$  and let the Stokes constant of  $F_0$  be non zero, then we can apply Theorem 3.2 to  $F_\mu$  to get the asymptotic series of the Lazutkin homoclinic invariant.

It remains to show is that the Stokes constant of  $G_\mu$  vanishes if and only if the Stokes constant of  $F_0$  vanishes. Let  $W_0^\pm(t)$  be parametrizations of the invariant manifolds of the map  $F_0$ , that satisfy the equation

$$W_0^\pm(t+1) = F_0(W_0^\pm(t))$$

and let  $V_0^\pm(t)$  be parametrizations of the invariant manifolds of the map  $G_0$ , that satisfy the equation

$$V_0^\pm(t+1) = G_0(V_0^\pm(t)).$$

Then  $V_0^\pm(t) = \Phi \circ W_0^\pm(t+c)$  for some  $c \in \mathbb{C}$ . We can choose the functions  $V_0^\pm$  to be real analytic and then  $c$  is a real constant. We define  $\delta_0(t) = W_0^+(t) - W_0^-(t)$  and we get

$$\begin{aligned} \omega\left(V_0^+(t) - V_0^-(t), \dot{V}_0^-(t)\right) &= \\ &= \omega\left(\Phi \circ W_0^+(t+c) - \Phi \circ W_0^-(t+c), \Phi' \circ W_0^-(t+c) \cdot \dot{W}_0^-(t+c)\right) \\ &= \omega\left(\Phi' \circ W_0^-(t+c) \cdot \delta_0(t+c) + O(\delta_0(t+c)^2), \Phi' \circ W_0^-(t+c) \cdot \dot{W}_0^-(t+c)\right) \\ &= \omega\left(\delta_0(t+c), \dot{W}_0^-(t+c)\right) + O(\delta_0(t+c)^2). \end{aligned}$$

We know that  $\delta_0(t) = O(e^{-2\pi i t})$ , so the term  $O(\delta_0(t+c)^2)$  vanishes with the limit. This gives

$$\begin{aligned}\theta_G &= \lim_{\text{Im } t \rightarrow -\infty} e^{2\pi i t} \omega \left( V_0^+(t) - V_0^-(t), \dot{V}_0^-(t) \right) \\ &= e^{-2\pi i c} \lim_{\text{Im } t \rightarrow -\infty} e^{2\pi i (t+c)} \omega \left( \delta_0(t+c), \dot{W}_0^-(t+c) \right) \\ &= e^{-2\pi i c} \theta_F.\end{aligned}$$

So we see that the absolute value of the Stokes constant is an invariant of the map  $G_0$ .  $\square$

## 4 Notation and outline of the proof

The proof of Theorem 3.2 takes up the rest of the paper, but before giving the proof it is useful to fix the notation.

We start with the family of maps  $f_\varepsilon$  and we define  $F_\varepsilon = f_\varepsilon^3$ . Since the map  $F_\varepsilon$  is analytic in  $x, y$  and  $\varepsilon$ , it can be decomposed in two ways, namely  $F_\varepsilon(x, y) = \sum_{n \geq 0} \varepsilon^n F_n(x, y)$  and  $F_\varepsilon(x, y) = \sum_{n \geq 1} \mathcal{F}_n(\varepsilon; x, y)$ . Here  $F_n$  are real analytic functions<sup>2</sup> independent of  $\varepsilon$  and  $\mathcal{F}_n$  are polynomials of degree  $n$  homogeneous in its three variables.

Given  $F_\varepsilon$ , we can construct  $\tilde{H}(\varepsilon; x, y)$  which is the formal Hamiltonian of the normal form. From this we can construct its formal time-1 flow  $\tilde{F}_\varepsilon$ . Similarly we define  $\tilde{F}_\varepsilon(x, y) = \sum_{n \geq 0} \varepsilon^n \tilde{F}_n(x, y)$  and  $\tilde{F}_\varepsilon(x, y) = \sum_{n \geq 1} \tilde{\mathcal{F}}_n(\varepsilon; x, y)$ . Notice that  $\tilde{\mathcal{F}}_n$  are still homogeneous polynomials.

The central objects in this analysis are the functions  $W^-(\varepsilon; \tau)$  and  $W^+(\varepsilon; \tau)$  which parametrize the stable and the unstable separatrix that are depicted as vertical in Figure 1. They satisfy the equation  $W^\pm(\varepsilon; \tau + 1) = F_\varepsilon(W^\pm(\varepsilon; \tau))$ . Unless it is explicitly stated, it will be assumed from now on that the separatrices are parametrized with step 1 as above. We fix the parametrization by asking that  $W^+(\varepsilon; 0)$  is the point where the stable separatrix meets the horizontal axis for the first time. Similarly  $W^-(\varepsilon; 0)$  is the point where the unstable separatrix meets the horizontal axis for the first time.

There are four formal solutions considered:  $\tilde{W}$ ,  $\tilde{\mathcal{W}}$ ,  $\tilde{\mathfrak{W}}$  and  $\tilde{\mathbb{W}}$ . The first,  $\tilde{W}$ , satisfies  $\tilde{W}(\varepsilon; \tau + 1) = F_\varepsilon(\tilde{W}(\varepsilon; \tau))$  and the second,  $\tilde{\mathcal{W}}$ , satisfies  $\tilde{\mathcal{W}}(\varepsilon; \tau + 1) = \tilde{F}_\varepsilon(\tilde{\mathcal{W}}(\varepsilon; \tau))$ . Both of those are formal series in  $\tanh(\frac{\varepsilon\tau}{2})$  and  $\varepsilon$ , so both have a singularity at  $\pi i$ . For the third we change the parametrization from  $\tau$  to  $t$  with  $t = \tau - \frac{\pi i}{\varepsilon}$ , then  $\tilde{\mathfrak{W}}$  is just  $\tilde{W}$  with  $\tanh(\frac{\varepsilon\tau}{2})$  expanded as Laurent series close to the singularity. Then  $\tilde{\mathfrak{W}}$  satisfies  $\tilde{\mathfrak{W}}(\varepsilon; t + 1) = F_\varepsilon(\tilde{\mathfrak{W}}(\varepsilon; t))$ . Finally  $\tilde{\mathbb{W}}$  is  $\tilde{\mathcal{W}}$  with  $\tanh(\frac{\varepsilon\tau}{2})$  expanded as Laurent series close to the singularity. The first component of  $\tilde{\mathcal{W}}$  and  $\tilde{\mathbb{W}}$  is even in  $\tau$  and  $t$  respectively and the second component is odd.

<sup>2</sup> There is an abuse of notation here since the subscript of  $F$  can denote either a real or a natural number. However it will be clear by the context which case is considered and for the case  $\varepsilon = 0$  both notations agree.

There are also two linear equation that play an important role to the proof. These are

$$U(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \cdot U(\varepsilon; \tau), \quad (4)$$

$$V(\varepsilon; \tau + 1) = D(\varepsilon; \tau) \cdot V(\varepsilon; \tau), \quad (5)$$

with

$$A(\varepsilon; \tau) = \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds$$

and

$$D(\varepsilon; \tau) = F'_\varepsilon(W^-(\varepsilon; \tau)).$$

Evidently  $\delta = W^+ - W^-$  satisfies the first one and  $\dot{W}^-$  satisfies the second. We denote by  $U$  the fundamental solution of the first and by  $V$  the fundamental solution of the second, normalized by  $\det U = \det V = 1$ . Moreover we have  $U \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta$  and  $V \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dot{W}^-$ .

We will see that there exists an open domain in variable  $\tau$  that contains the origin and goes close to the singularities  $\pm \frac{\pi}{\varepsilon} i$  in which both  $U$  and  $V$  are analytic and their difference is small.

If we look at equations (4) and (5) formally we see that the formal matrices  $\tilde{A}$  and  $\tilde{D}$  coincide. The formal equation close to the singularity is

$$\tilde{V}(\varepsilon; t + 1) = \tilde{D}(\varepsilon; t) \cdot \tilde{V}(\varepsilon; t).$$

We denote by  $\tilde{V}$  its fundamental solution that satisfies  $\det \tilde{V} = 1$  and  $\tilde{V} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dot{\tilde{W}}$ .

To prove Theorem 3.2, first we prove the existence of the formal solution  $\tilde{\mathcal{W}}$ . Then we prove that the formal solution close to the singularity approximates the separatrix close to the singularity. The process to do so is called complex matching, see [GL01].

Then we introduce the function

$$\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)).$$

We will see that this function is approximately periodic. This enables us to compute its “first Fourier coefficient”. The derivative of this function at a homoclinic point gives the homoclinic invariant. We estimate the value of this function close to the singularity where it is polynomially small with  $\varepsilon$  and finally we translate the result to the real axis where we see it is exponentially small.

Notice that throughout the proof we treat  $\varepsilon_0$  as if it was fixed but we are allowed to decrease it if the need arises. We also need to choose  $\Lambda > 1$  such that  $\Lambda^2 \varepsilon_0 < 1$ . We are also allowed to increase  $\Lambda$  if the need arises making sure that  $\varepsilon_0$  will be decreased proportionately. The constant  $|Lambda$  is used to fine tune the domains where the separatrices can be approximated by the formal solution.

## 5 Formal solutions

### 5.1 Formal separatrix close to the saddle point

By Proposition 2.2 we know that there is a formal change of coordinates that transforms the map  $F_\mu$  to the 1-flow of a formal Hamiltonian  $\tilde{H}(\varepsilon; x, y)$ . By solving formally Hamilton's equations we have the following lemma.

**Lemma 5.1.** *Let  $\sigma = \tanh(\frac{\varepsilon\tau}{2})$  and let  $\tilde{H}(\mu(\varepsilon); x, y)$  be a formal Hamiltonian as described in Proposition 2.2. Then there exist a real formal power series such that  $\mu(\varepsilon) = \sum_{n \geq 1} \mu_n \varepsilon^n$  and a real formal solution  $\tilde{\mathcal{W}}(\varepsilon; \tau) = (\tilde{x}(\varepsilon; \tau), \tilde{y}(\varepsilon; \tau))$  of Hamilton's equations<sup>3</sup>*

$$\begin{aligned}\dot{x} &= \partial_y \tilde{H}(\mu(\varepsilon); x, y), \\ \dot{y} &= -\partial_x \tilde{H}(\mu(\varepsilon); x, y),\end{aligned}$$

such that

$$\begin{aligned}\tilde{x}(\varepsilon; \tau) &= \sum_{n \geq 1} \varepsilon^n P_n(\sigma), \\ \tilde{y}(\varepsilon; \tau) &= \sum_{n \geq 1} \varepsilon^n Q_n(\sigma),\end{aligned}$$

with  $P_n(\sigma)$  even polynomials of degree  $2\lfloor \frac{n}{2} \rfloor$ ,  $Q_n(\sigma)$  odd polynomials of degree  $2\lfloor \frac{n+1}{2} \rfloor - 1$  and  $P_1(\sigma) = \frac{1}{2\sqrt{3}b_{0,0}}$ ,  $Q_1(\sigma) = \frac{\sigma}{2b_{0,0}}$ ,  $\mu_1 = \frac{1}{2\sqrt{3}a_{0,1}}$ . Moreover  $P_n$ ,  $Q_n$  and  $\mu_n$  depend uniquely on  $P_1$ ,  $Q_1$  and  $\mu_1$  for all  $n > 1$ .

*Proof.* Note that  $\tilde{H}$  is invariant under the transformation  $(x, y) \mapsto (x, -y)$ . So we choose a power series with each degree having the first component even and the second odd.

To solve Hamilton's equations we use the fact that  $\dot{\sigma} = \frac{1}{2}\varepsilon(1 - \sigma^2)$ . Then it is a matter of substitution and gathering of terms in increasing degrees of  $\varepsilon$ .

The first term that appear in Hamilton's equations is of order 2, let  $P_1(\sigma) = A_{1,0}$  and  $Q_1(\sigma) = A_{1,1}\sigma$ . Then we have

$$\begin{aligned}0 &= 2b_{0,0}A_{1,0}A_{1,1}\sigma\varepsilon - 2a_{0,1}\mu_1A_{1,1}\sigma a_{0,1}\mu_1\varepsilon^2 \\ \frac{1}{2}A_{1,1}(1 - \sigma^2)\varepsilon^2 &= b_{0,0}(A_{1,0}^2 - A_{1,1}^2\sigma^2)\varepsilon^2 + 2a_{0,1}\mu_1A_{1,0}\varepsilon^2\end{aligned}$$

and from the possible solutions we choose  $A_{1,0} = \frac{1}{2\sqrt{3}b_{0,0}}$ ,  $A_{1,1} = \frac{1}{2b_{0,0}}$  and  $\mu_1 = \frac{1}{2\sqrt{3}a_{0,1}}$ .

Then we let

$$P_n(\sigma) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A_{n,2k} \sigma^{2k},$$

---

<sup>3</sup>Here the dot denotes derivation with respect to  $\tau$ .

$$Q_n(\sigma) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A_{n,2k+1} \sigma^{2k+1}.$$

Thus for each  $n$  there are  $n + 2$  coefficients, counting  $\mu_n$  as an unknown. By taking into the account that at the power  $\epsilon^n$ ,  $P_n$  and  $Q_n$  appear only in the second order terms of the Hamiltonian equations, we find that we need to solve a linear system. We have two cases.

- $n = 2m$

We arrange the unknowns by  $(\mu_{2m}, A_{2m,1}, \dots, A_{2m,2m-1}, A_{2m,0}, \dots, A_{2m,2m})$ . Then the matrix,  $M$ , of the system is of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $A, B, C, D$  being  $(m+1) \times (m+1)$  matrices:

$$A = \begin{pmatrix} d_0 & t_1 & & & & \\ & d_1 & t_2 & & & \\ & & d_2 & t_3 & & \\ & & & \ddots & \ddots & \\ & & & & d_{m-1} & t_m \\ & & & & & d_m \end{pmatrix},$$

$$B = -\frac{2}{\sqrt{3}} \text{Id}_{n+1},$$

$$C = \begin{pmatrix} \frac{a_{0,1}}{b_{0,0}} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} -1 & 1 & & & & \\ & -2 & 2 & & & \\ & & -3 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & -m & m \\ & & & & & -m-1 \end{pmatrix},$$

with  $d_0 = \frac{a_{0,1}}{\sqrt{3}b_{0,0}}$  and for  $j > 0$   $d_j = \frac{1}{2} - j$ ,  $t_j = \frac{1}{2} + j$ .

- $n = 2m + 1$

We arrange the unknowns by  $(\mu_{2m+1}, A_{2m+1,1}, \dots, A_{2m+1,2m+1}, A_{2m+1,0}, \dots, A_{2m+1,2m})$ . Then the matrix,  $M$ , of the system has a similar form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $A, B, C, D$  are  $(m+2) \times (m+2)$ ,  $(m+2) \times (m+1)$ ,  $(m+1) \times (m+2)$  and  $(m+1) \times (m+1)$  matrices respectively and

$$A = \begin{pmatrix} d_0 & t_1 & & & & \\ & d_1 & t_2 & & & \\ & & d_2 & t_3 & & \\ & & & \ddots & \ddots & \\ & & & & d_m & t_{m+1} \\ & & & & & d_{m+1} \end{pmatrix},$$

$$B = \begin{pmatrix} -\frac{2}{\sqrt{3}} \text{Id}_{n+1} \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{a_{0,1}}{b_{0,0}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 1 & & & & \\ & -2 & 2 & & & \\ & & -3 & 3 & & \\ & & & \ddots & \ddots & \\ & & & & -m & m \\ & & & & & -m-1 \end{pmatrix},$$

Then we have<sup>4</sup>  $\det(M) = \det(A - BD^{-1}C) \det(D)$ . Since

$$D^{-1} = - \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & & \frac{1}{3} & \cdots & \frac{1}{m} & \frac{1}{m+1} \\ & & & \ddots & \vdots & \vdots \\ & & & & \frac{1}{m} & \frac{1}{m+1} \\ & & & & & \frac{1}{m+1} \end{pmatrix}$$

we get

$$BD^{-1}C = \begin{pmatrix} \frac{2a_{0,1}}{\sqrt{3}b_{0,0}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

so  $\det(A - BD^{-1}C) \neq 0$ . This means that the matrix is invertible so the system is solvable.  $\square$

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<sup>4</sup> This is a direct implication of the equality

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I_m \end{pmatrix} \cdot \begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}.$$



**Lemma 5.2.** *The formal solution  $\tilde{\mathcal{W}}(\varepsilon; \tau)$  satisfies  $\tilde{\mathcal{W}}(\varepsilon; \tau + 1) = \tilde{F}_\varepsilon(\tilde{\mathcal{W}}(\varepsilon; \tau))$ .*

*Proof.* Let  $\mathcal{H}$  be a formal series in  $\varepsilon$ ,  $x$  and  $y$  and  $\mathcal{X}$  a formal series in  $\varepsilon$  and  $\tanh(\frac{\varepsilon\tau}{2})$ . Let  $T : (\mathcal{H}, \mathcal{X}) \mapsto \mathcal{X}(\tau + 1) - \phi_{\mathcal{H}}^1(\mathcal{X}(\tau))$ .

Then  $T(\mathcal{H}, \mathcal{X})$  is a power series in  $\tanh(\frac{\varepsilon\tau}{2})$  and  $\varepsilon$ . However at each degree of  $\varepsilon$  we have a polynomial in  $\tanh(\frac{\varepsilon\tau}{2})$ . This allows us to treat  $T(\mathcal{H}, \mathcal{X})$  as if it was a formal power series only in  $\varepsilon$  and define the standard valuation and the corresponding metric. The map  $T$  is continuous with the induced topology.

Let  $\tilde{H}_n$  be the truncation of  $\tilde{H}(\mu(\varepsilon); x, y)$  to power  $n$  and  $\tilde{\mathcal{W}}_{\tilde{H}_n}$  the separatrix. Since  $\tilde{H}_n$  is a polynomial,  $\tilde{\mathcal{W}}_{\tilde{H}_n}$  is convergent and then  $T(\tilde{H}_n, \tilde{\mathcal{W}}_{\tilde{H}_n}) = 0$ . Then taking the limit  $n \rightarrow \infty$  we get  $T(\tilde{H}(\mu(\varepsilon); x, y), \tilde{\mathcal{W}}) = 0$  by continuity.  $\square$

We denote  $\mathcal{Z}_n(\tau) = (P_n(\sigma), Q_n(\sigma))$ , so  $\tilde{\mathcal{W}}(\varepsilon; \tau) = \sum_{n \geq 1} \varepsilon^n \mathcal{Z}_n(\sigma)$ , and  $\tilde{\mathcal{Z}}_n(\varepsilon; \tau) = \sum_{m=1}^n \varepsilon^m \mathcal{Z}_m(\sigma)$ .

**Corollary 5.3.** *Let  $F_\varepsilon$  be a map that agrees with  $\tilde{F}_\varepsilon$  up to degree  $n$ . Then we have  $\tilde{\mathcal{Z}}_n(\varepsilon; \tau + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; \tau)) = O(\varepsilon^{n+2})$ .*

*Proof.* This is derived directly from the continuity of the map  $T$  defined above.  $\square$

## 5.2 Approximation of the separatrix

For the existence and the difference of the two separatrices we have the following theorem.

**Theorem 5.4.** *Let  $\varepsilon \in (0, \varepsilon_0)$  for some  $\varepsilon_0 > 0$  and let  $\Gamma_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon H_\varepsilon(\mathbf{x})$  denote a real analytic family of area preserving maps, that is also analytic in  $\varepsilon$ , defined on a bounded domain  $\mathcal{D} \subset \mathbb{C}^2$  for all  $\varepsilon$ . Moreover let the origin be a hyperbolic fixed point for every map and  $\varepsilon$  be the logarithm of the largest eigenvalue. We consider the separatrix equation*

$$\mathbf{X}^-(\varepsilon; s + \varepsilon) = \Gamma_\varepsilon(\mathbf{X}^-(\varepsilon; s)).$$

*Then the following are true.*

- *The separatrix equation has a solution tangent to the eigenvector of  $\Gamma'_\varepsilon(0)$  that corresponds to the eigenvalue that is bigger than 1.*
- *There exists a formal solution of the separatrix equation of the form*

$$\tilde{\mathbf{X}}(\varepsilon; s) = \sum_{k \geq 0} \varepsilon^k \Psi_k(e^s),$$

*with  $\Psi_k$  being analytic around 0 and  $\Psi_k(0) = 0$ .*

- *Let  $\tilde{\mathbf{X}}_n(\varepsilon; s) = \sum_{k=0}^{n-1} \varepsilon^k \Psi_k(e^s)$ . Then we have*

$$\left| \mathbf{X}^-(\varepsilon; s) - \tilde{\mathbf{X}}_n(\varepsilon; s) \right| \leq C_n \varepsilon^n$$

*for all  $s \in D$ , where  $D$  is the domain on which all  $\Psi_k$  are bounded.*

For a proof of this theorem see [BG08].

In order to apply the theorem we need to scale and translate the map. Let  $\varepsilon w_*$  be one equilibrium point, namely  $\varepsilon w_* = F_\varepsilon(\varepsilon w_*)$ . We define the map

$$G_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon} F_\varepsilon(\varepsilon(\mathbf{x} + w_*)) - w_*.$$

We see that  $G_\varepsilon(0) = 0$  and that  $\mathbf{X}^-(\varepsilon; s) = \frac{1}{\varepsilon} W^-(\varepsilon; \frac{s}{\varepsilon}) - w_*$  satisfies both

$$\begin{aligned} \mathbf{X}^-(\varepsilon; s + \varepsilon) &= G_\varepsilon(\mathbf{X}^-(\varepsilon; s)) \\ \lim_{s \rightarrow -\infty} \mathbf{X}^-(\varepsilon; s) &= 0. \end{aligned}$$

Moreover by defining  $\tilde{G}_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon} \tilde{F}_\varepsilon(\varepsilon(\mathbf{x} + w_*)) - w_*$ , we see that we get a formal solution of the separatrix by defining

$$\begin{aligned} \tilde{\mathbf{X}}(\varepsilon; s) &= \frac{1}{\varepsilon} \tilde{W}(\varepsilon; \frac{s}{\varepsilon}) = \sum_{n \geq 1} \varepsilon^{n-1} \mathcal{Z}_n \left( \tanh \left( \frac{s}{2} \right) \right) - w_* \\ &= \sum_{n \geq 1} \varepsilon^{n-1} \mathcal{Z}_n \left( \frac{e^s - 1}{e^s + 1} \right) - w_*. \end{aligned}$$

We know that in our case the asymptotic can be written as a series of polynomials in  $\tanh(s/2)$ , so let  $D$  be a domain where  $\tanh(s/2)$  is bounded. This means that each term of the asymptotic is bounded. From the above we see that if  $F_\varepsilon$  agrees with  $\tilde{F}_\varepsilon$  at least up to degree  $n+1$  we can apply the theorem and translate the result back to our original setting to get that there exists  $C_n > 0$  such that for all  $t \in D$  it holds

$$\left| W^-(\varepsilon; \tau) - \tilde{\mathcal{Z}}_n(\varepsilon; \tau) \right| \leq C_n \varepsilon^{n+1}. \quad (6)$$

### 5.3 Formal separatrix close to the singularity

We saw, using Theorem 5.4, that there exists a formal solution for the separatrix equation and it can be made to agree with  $\tilde{\mathcal{W}}$  up to any order. Let  $\tilde{W}$  denote this formal solution, i.e.  $\tilde{W}$  satisfies formally  $\tilde{W}(\varepsilon; \tau + 1) = F_\varepsilon(\tilde{W}(\varepsilon; \tau))$ . Note that unlike  $\tilde{\mathcal{W}}$ ,  $\tilde{W}$  does not have one even and one odd component. However the two solutions are conjugated by a power series so we deduce that  $\tilde{W}$  is in every order of  $\varepsilon$  is an analytic function of  $\tanh(\varepsilon\tau)/2$ .

Both  $\tilde{W}$  and  $\tilde{\mathcal{W}}$  have a singularity at  $\pi i/\varepsilon$  as the hyperbolic tangent has a simple pole there. We introduce a new parameter  $t$  by translating the origin at the singularity, so  $\tau = t + \pi i/\varepsilon$ . Now we can take the Laurent series around the origin. Since the power of  $\sigma$  in  $P_n$  and  $Q_n$  is at most  $n$ , the expansion does not have terms with negative powers of  $\varepsilon$ . The monomials that appear in this expansion are summarized in Table 1. This is valid for both  $\tilde{W}$  and  $\tilde{\mathcal{W}}$ .

We expand  $\varepsilon^n \mathcal{Z}_n(\varepsilon; t + \frac{\pi}{\varepsilon} i) = \sum_{k \geq 0} W_{n,k} \varepsilon^k t^{k-n}$ . This denotes both components, so  $W_{n,k}$  should be thought of as a point in  $\mathbb{C}^2$ . On Table 1 each row shows the

	$\tilde{W}_0$	$\varepsilon \tilde{W}_1$	$\varepsilon^2 \tilde{W}_2$	$\varepsilon^3 \tilde{W}_3$	$\varepsilon^4 \tilde{W}_4$	$\varepsilon^5 \tilde{W}_5$	$\varepsilon^6 \tilde{W}_6$	$\varepsilon^7 \tilde{W}_7$	$\dots$
$\varepsilon \mathcal{Z}_1$	$t^{-1}$	$\varepsilon$	$\varepsilon^2 t$	$\varepsilon^3 t^2$	$\varepsilon^4 t^3$	$\varepsilon^5 t^4$	$\varepsilon^6 t^5$	$\varepsilon^7 t^6$	$\dots$
$\varepsilon^2 \mathcal{Z}_2$	$t^{-2}$	$\varepsilon t^{-1}$	$\varepsilon^2$	$\varepsilon^3 t$	$\varepsilon^4 t^2$	$\varepsilon^5 t^3$	$\varepsilon^6 t^4$	$\varepsilon^7 t^5$	$\dots$
$\varepsilon^3 \mathcal{Z}_3$	$t^{-3}$	$\varepsilon t^{-2}$	$\varepsilon^2 t^{-1}$	$\varepsilon^3$	$\varepsilon^4 t$	$\varepsilon^5 t^2$	$\varepsilon^6 t^3$	$\varepsilon^7 t^4$	$\dots$
$\varepsilon^4 \mathcal{Z}_4$	$t^{-4}$	$\varepsilon t^{-3}$	$\varepsilon^2 t^{-2}$	$\varepsilon^3 t^{-1}$	$\varepsilon^4$	$\varepsilon^5 t$	$\varepsilon^6 t^2$	$\varepsilon^7 t^3$	$\dots$
$\varepsilon^5 \mathcal{Z}_5$	$t^{-5}$	$\varepsilon t^{-4}$	$\varepsilon^2 t^{-3}$	$\varepsilon^3 t^{-2}$	$\varepsilon^4 t^{-1}$	$\varepsilon^5$	$\varepsilon^6 t$	$\varepsilon^7 t^2$	$\dots$
$\varepsilon^6 \mathcal{Z}_6$	$t^{-6}$	$\varepsilon t^{-5}$	$\varepsilon^2 t^{-4}$	$\varepsilon^3 t^{-3}$	$\varepsilon^4 t^{-2}$	$\varepsilon^5 t^{-1}$	$\varepsilon^6$	$\varepsilon^7 t$	$\dots$
$\varepsilon^7 \mathcal{Z}_7$	$t^{-7}$	$\varepsilon t^{-6}$	$\varepsilon^2 t^{-5}$	$\varepsilon^3 t^{-4}$	$\varepsilon^4 t^{-3}$	$\varepsilon^5 t^{-2}$	$\varepsilon^6 t^{-1}$	$\varepsilon^7$	$\dots$
$\varepsilon^8 \mathcal{Z}_8$	$t^{-8}$	$\varepsilon t^{-7}$	$\varepsilon^2 t^{-6}$	$\varepsilon^3 t^{-5}$	$\varepsilon^4 t^{-4}$	$\varepsilon^5 t^{-3}$	$\varepsilon^6 t^{-2}$	$\varepsilon^7 t^{-1}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 1: Monomials in expansion close to the singularity

monomials in the expansion of  $\varepsilon^n \mathcal{Z}_n$  without the coefficients. By changing summation order we can sum by columns so we have

$$\tilde{W}(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}) = \sum_{n \geq 0} \varepsilon^n \tilde{W}_n(t),$$

with each  $\tilde{W}_n(t)$  being a 2-vector of formal series in  $t$ . From now on  $\tilde{\mathfrak{W}}(\varepsilon; t)$  will denote  $\tilde{W}(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i})$  summed by columns. We substitute the series in the equation  $\tilde{\mathfrak{W}}(\varepsilon; t+1) = F_\varepsilon(\tilde{\mathfrak{W}}(\varepsilon; t))$  and we gather terms in powers of  $\varepsilon$  and we get

$$\begin{aligned} n=0 : \quad & \tilde{W}_0(t+1) = F_0(\tilde{W}_0(t)) \\ n>0 : \quad & \tilde{W}_n(t+1) = F'_0(\tilde{W}_0(t)) \cdot \tilde{W}_n(t) + B_n(t), \end{aligned} \quad (7)$$

with  $B_n(t)$  depending on  $\tilde{W}_m$  and  $F_m$ ,  $0 \leq m < n$ .

## 5.4 Borel transform and linear difference equations

For the resonant map, the construction of the solution by the asymptotic was done using the Borel-Laplace resummation method. For the Laplace transform it holds that

$$\mathcal{L}[s^n](t) = \frac{n!}{t^{n+1}}.$$

The Borel transform is defined as the formal inverse of the Laplace, namely

$$\mathcal{B}[t^{-n-1}] = \frac{s^n}{n!}.$$

This means that the Borel transform of a divergent series can be convergent. If this is the case and if the Borel transform can be extended beyond a neighbourhood of the origin, its Laplace transform will give the Borel-Laplace sum of the initial formal series. This method was generalized with the theory of resurgent functions in [Éca81]. For an elementary introduction to the theory tailored to the present problem, see [GM17]. For the purpose of the present text we will use the term *resurgent function* to refer to a function with a singularity at the

origin which is a logarithmic branching plus a pole, on all sheets the points  $2\pi i\mathbb{Z}$  are branching singularities and is of exponential type along paths that avoid the set  $2\pi i\mathbb{Z}$  and eventually go to infinity following a straight non-vertical line. The formal series that is the formal Laplace transform of a resurgent function will be called *resurgent series*.

The Borel transform maps the product of two series to the convolution of their Borel sums. This means that if we consider the space of all resurgent series as a ring by multiplication, the space of their Borel sums is a ring under convolution and the Borel transform acts as a ring homomorphism. On the space of the Borel sums a set of new operators that act as derivations can be defined. These derivatives, the celebrated *alien derivatives*, can be pulled back on the space of resurgent series and do not have a classical counterpart.

It was proven in [GM17] that  $\tilde{W}_0$  is a resurgent series and that using Borel-Laplace summation we can construct the functions  $W_0^+$  and  $W_0^-$ . The function  $W_0^+$  is analytic on  $\mathbb{C}$  minus a sector containing the negative real semi axis and  $W_0^-$  is analytic on  $\mathbb{C}$  minus a sector containing the positive real semi axis. These domains are called sectorial neighbourhoods of infinity. Using that, we get the asymptotic of their difference, which is described in theorem 3.1. It was also proved that there exists a  $2 \times 2$  matrix of resurgent series  $\tilde{\mathcal{V}}_0 \in \mathbb{C}[t][[t^{-1}]]^{2 \times 2}$  such that it satisfies the variational equation (3) and  $\det \tilde{\mathcal{V}}_0(t) = 1$ .

We are interested in solutions of equations of the form

$$X(t+1) = A(t) \cdot X(t) + B(t) \quad (8)$$

with  $A(t) = F'_0(\tilde{W}_0(t))$  and  $B \in \mathbb{C}[t][[t^{-1}]]^2$ . We define  $X(t) = \tilde{\mathcal{V}}(t) \cdot Y(t)$  and from this we get

$$Y(t+1) - Y(t) = \tilde{\mathcal{V}}^{-1}(t) \cdot A^{-1}(t) \cdot B(t). \quad (9)$$

Then  $Y \in \mathbb{C}[t][[t^{-1}]]^2$  if and only if the formal series in the matrix  $\tilde{\mathcal{V}}^{-1} \cdot A^{-1} \cdot B$  do not contain the term  $\frac{1}{t}$ . If the term  $\frac{1}{t}$  appears then  $Y$  has also logarithmic terms.

Let  $\hat{A}$  denote the Borel transform of  $A$ . Since  $F_0$  is convergent and  $\tilde{W}_0$  is a resurgent series,  $\hat{A}$  is also a resurgent function, for a proof of this see [Sau13]. Suppose that the Borel transform of  $B$  is a resurgent function  $\hat{B}$ , then the Borel transform of equation (8) is

$$e^{-s} \hat{X}(s) = \hat{A} * \hat{X}(s) + \hat{B}(s) \quad (10)$$

and by defining  $\hat{X}(s) = \hat{\mathcal{V}} * \hat{Y}(s)$  we get derive the equation

$$e^{-s} \hat{Y}(s) - \hat{Y}(s) = \hat{\mathcal{V}}^{-1} * \hat{A}^{-1} * B(s). \quad (11)$$

This last equation can be solved trivially and gives

$$\hat{Y}(s) = \frac{e^s}{1 - e^s} \left( \hat{\mathcal{V}}^{-1} * \hat{A}^{-1} * B(s) \right)$$

so finally we get

$$\hat{X}(s) = \hat{\mathcal{V}} * \left( \frac{e^s}{1 - e^s} \left( \hat{\mathcal{V}}^{-1} * \hat{A}^{-1} * B \right) \right)(s).$$

From this we deduce that  $\hat{X}$  is a resurgent function.

We can now apply the above at equation (7). In this case  $B_n$  depends on  $F_m$  and  $\tilde{W}_m$  with  $m < n$ . Then using the results of [Sau15] it can be shown inductively that the Borel transform of any  $\tilde{W}_n$  defines a resurgent function  $\hat{W}_n$ . Moreover it can be shown that for all  $n \in \mathbb{N}$  it holds  $\hat{W}_n \in \mathbb{C}[t][[t^{-1}]]^2$ . This can be done by supposing that for some  $k \in \mathbb{N}$ ,  $\tilde{W}_k$  contains logarithmic terms, then by transforming  $F_\varepsilon$  to agree with the normal form up to order  $k+1$  and matching it with the formal solution we deduce that  $\hat{W}_k$  does not contain logarithmic term, so such  $k$  cannot exist.

Since the Borel transform  $\hat{W}_n$  is resurgent, there are two Borel-Laplace sums for each  $\tilde{W}_n$ , namely  $W_n^+$  and  $W_n^-$  and each one is the sum of a polynomial of at most degree  $n$  and a function decaying as  $t^{-1}$  as  $t$  goes to infinity. Both  $W_n^+$  and  $W_n^-$  are analytic in the sectorial neighbourhoods of infinity in which  $W_0^+$  and  $W_0^-$  are defined.

## 5.5 Formal solution to the variational equation

Let  $\tilde{\mathbb{W}}(\varepsilon; t)$  be the formal separatrix expanded in powers of  $\varepsilon$  and  $t$ . We define a degree of each monomial by  $\deg(\varepsilon^n t^m) = 2n - m$ . We know that  $\tilde{\mathbb{W}}$  satisfies the equation

$$\begin{aligned}\partial_t \tilde{\mathbb{W}}_1(\varepsilon; t) &= \tilde{H}_y(\varepsilon; \tilde{\mathbb{W}}_1(\varepsilon; t), \tilde{\mathbb{W}}_2(\varepsilon; t)) \\ \partial_t \tilde{\mathbb{W}}_2(\varepsilon; t) &= -\tilde{H}_x(\varepsilon; \tilde{\mathbb{W}}_1(\varepsilon; t), \tilde{\mathbb{W}}_2(\varepsilon; t)),\end{aligned}$$

with  $\tilde{H}_x(\varepsilon; x, y) = \partial_x \tilde{H}(\varepsilon; x, y)$  and  $\tilde{H}_y(\varepsilon; x, y) = \partial_y \tilde{H}(\varepsilon; x, y)$ . We also define  $\tilde{H}_{xy}(\varepsilon; x, y) = \partial_x \partial_y \tilde{H}(\varepsilon; x, y)$  and similarly  $\tilde{H}_{xx}(\varepsilon; x, y)$  and  $\tilde{H}_{yy}(\varepsilon; x, y)$ .

We know that  $\tilde{H}(\varepsilon; x, -y) = \tilde{H}(\varepsilon; x, y)$ . This implies that

$$\begin{aligned}\tilde{H}_x(\varepsilon; x, -y) &= \tilde{H}_x(\varepsilon; x, y), \\ \tilde{H}_y(\varepsilon; x, -y) &= -\tilde{H}_y(\varepsilon; x, y), \\ \tilde{H}_{xx}(\varepsilon; x, -y) &= \tilde{H}_{xx}(\varepsilon; x, y), \\ \tilde{H}_{xy}(\varepsilon; x, -y) &= -\tilde{H}_{xy}(\varepsilon; x, y), \\ \tilde{H}_{yy}(\varepsilon; x, -y) &= \tilde{H}_{yy}(\varepsilon; x, y).\end{aligned}$$

We also define

$$\tilde{H}_\kappa(t) := \tilde{H}_\kappa(\varepsilon; \tilde{\mathbb{W}}_1(\varepsilon; t), \tilde{\mathbb{W}}_2(\varepsilon; t))$$

with  $\kappa \in \{x, y, xx, xy, yy\}$ . Notice that the dependence on  $\varepsilon$  is implied.

Since  $\tilde{\mathbb{W}}_1(\varepsilon; t)$  is even in  $t$  and  $\tilde{\mathbb{W}}_2(\varepsilon; t)$  is odd in  $t$  we have that

- $\tilde{H}_x(t)$ ,  $\tilde{H}_{xx}(t)$  and  $\tilde{H}_{yy}(t)$  are even functions of  $t$ ,
- $\tilde{H}_y(t)$  and  $\tilde{H}_{xy}(t)$  are odd functions of  $t$ .

Let  $\tilde{\mathbb{V}}(\varepsilon; t)$  be the fundamental solution of the variation of the above equation, i.e.

$$\partial_t \tilde{\mathbb{V}}(\varepsilon; t) = \tilde{J}(t) \tilde{\mathbb{V}}(\varepsilon; t)$$

with

$$\tilde{J}(t) = \begin{pmatrix} \tilde{H}_{xy}(t) & \tilde{H}_{yy}(t) \\ -\tilde{H}_{xx}(t) & -\tilde{H}_{xy}(t) \end{pmatrix}.$$

Let  $\tilde{\mathbb{V}}(\varepsilon; t) = (\tilde{\Xi}(\varepsilon; t), \tilde{Z}(\varepsilon; t))$ . We ask that  $\det \tilde{\mathbb{V}}(\varepsilon; t) = 1$  and that  $\tilde{Z}(\varepsilon; t) = \partial_t \tilde{\mathbb{W}}(\varepsilon; t)$ . We write

$$\tilde{\Xi}_1(\varepsilon; t) = \frac{1 + \tilde{\Xi}_2(\varepsilon; t)\tilde{Z}_1(\varepsilon; t)}{\tilde{Z}_2(\varepsilon; t)}$$

and we substitute this in the variational equation to get

$$\partial_t \tilde{\Xi}_2(\varepsilon; t) = - \left( \tilde{H}_{xx}(t) \frac{\tilde{Z}_1(\varepsilon; t)}{\tilde{Z}_2(\varepsilon; t)} + \tilde{H}_{xy}(t) \right) \tilde{\Xi}_2(\varepsilon; t) - \frac{\tilde{H}_{xx}(t)}{\tilde{Z}_2(\varepsilon; t)}.$$

Since  $\tilde{Z}_2(\varepsilon; t)$  is a solution for the homogeneous equation we write  $\tilde{\Xi}_2(\varepsilon; t) = C(\varepsilon; t)\tilde{Z}_2(\varepsilon; t)$  and substitute in the previous equation to finally get

$$\partial_t C(\varepsilon; t) = - \frac{\tilde{H}_{xx}(t)}{\tilde{Z}_2(\varepsilon; t)^2}.$$

Since both  $\tilde{H}_{xx}(t)$  and  $\tilde{Z}_2(\varepsilon; t)$  are even series the above equation can be solved in the space of power series and  $C(\varepsilon; t)$  is an odd series without logarithmic terms. This implies that  $\tilde{\Xi}_2(\varepsilon; t)$  is odd and  $\tilde{\Xi}_1(\varepsilon; t)$  is even.

This formal solution also satisfies the variational equation of the normal form map, i.e.

$$\tilde{\mathbb{V}}(\varepsilon; t+1) = \tilde{F}'_\varepsilon(\tilde{\mathbb{W}}(\varepsilon; t)) \cdot \tilde{\mathbb{V}}(\varepsilon; t).$$

We write

$$\tilde{\mathbb{V}}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \tilde{\mathbb{V}}_n(t)$$

and we see that  $\tilde{\mathbb{V}}_0(t)$  satisfies the equation

$$\tilde{\mathbb{V}}_0(t+1) = \tilde{F}'_0(\tilde{\mathbb{W}}_0(t)) \cdot \tilde{\mathbb{V}}_0(t).$$

So  $\tilde{\mathbb{V}}$  is the fundamental solution of the the variational equation of the normal form map at resonance.

Then for  $n > 0$  we get

$$\tilde{\mathbb{V}}_n(t+1) = F'_0(\tilde{\mathbb{W}}_0(t))\tilde{\mathbb{V}}_n(t) + \tilde{B}_n(t)$$

with  $\tilde{B}_n$  depending on  $\tilde{\mathbb{V}}_m$ 's with  $m < n$ . As we saw in the previous section these equations define resurgent series. Since we know that the series contain only integer powers of  $t$ , then  $\hat{\mathbb{V}}_n$ , the Borel transform of  $\tilde{\mathbb{V}}_n$ , is a simple resurgent function. By looking at the equation above we get that for  $\tilde{\mathbb{V}}_n$  the biggest power of  $t$  is  $n+2$ .

Now that we know that  $\tilde{W}_n$  are resurgent we can define the action of the alien derivative  $\Delta_{2\pi i}$  on  $\tilde{\mathbb{W}}$  by

$$\Delta_{2\pi i}[\tilde{\mathbb{W}}](\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \Delta_{2\pi i}[\tilde{W}_n](t).$$

Since  $\Delta_{2\pi\mathbf{i}}$  satisfies the Leibniz rule,<sup>5</sup> then all  $\Delta_{2\pi\mathbf{i}}[\tilde{W}](\varepsilon; t)$  satisfy the variational equation, which means that for all  $m \geq 1$  there are constants  $\Theta_{2\pi\mathbf{i}}$  and  $q_{2\pi\mathbf{i}}$  such that

$$\Delta_{2\pi\mathbf{i}}[\tilde{W}](\varepsilon; t) = \Theta_{2\pi\mathbf{i}}\tilde{\Xi}(\varepsilon; t) + q_{2\pi\mathbf{i}}\tilde{Z}(\varepsilon; t).$$

## 6 Complex matching

In this section we will show that the formal solution  $\tilde{\mathfrak{W}}^\pm$  describes the asymptotic behaviour of  $W^\pm$  close to the singularity  $\pi\mathbf{i}/\varepsilon$ .

Let  $\text{SQ}(r), \text{SQ}_{+1}(r), \text{HP}(r) \in \mathbb{C}$  be defined as follows

$$\begin{aligned} \text{SQ}(r) &:= \{z \in \mathbb{C} : |\text{Im}(z)| < r, \text{Re}(z) > -r\}, \\ \text{SQ}_{+1}(r) &:= \{z \in \mathbb{C} : |\text{Im}(z)| < r, \text{Re}(z) > -(r+1)\}, \\ \text{HP}(r) &:= \{z \in \mathbb{C} : \text{Re}(z) > r\}. \end{aligned}$$

Recall that we assume that there exists  $\varepsilon_0 > 0$  such that  $\varepsilon \in (0, \varepsilon_0)$ . Since we are interested in the asymptotic behaviour of the separatrices, we can choose  $\varepsilon_0$  to be as small as it is convenient. We choose  $\Lambda > 1$  such that  $\Lambda^2\varepsilon_0 < 1$ . During the course of this proof we will see that it may be important to increase the value of  $\Lambda$ . This is not a problem since we can simultaneously decrease  $\varepsilon_0$  such that the relation  $\Lambda^2\varepsilon_0 < 1$  still holds. So  $\Lambda$  should be thought of as a constant that can be tuned.

We choose  $\Lambda$ , fix  $R > 1$  and we define the following domains:

$$\begin{aligned} D_0 &:= \mathbb{C} \setminus (\text{SQ}((\Lambda\varepsilon)^{-1}) \cup \text{HP}(R)), \\ D_1 &:= \text{SQ}_{+1}((\Lambda\varepsilon)^{-1}) \setminus (\text{SQ}(\varepsilon^{-\frac{1}{2}}) \cup \text{HP}(\Lambda)), \\ D_2 &:= \text{SQ}_{+1}(\varepsilon^{-\frac{1}{2}}) \setminus (\text{SQ}(\Lambda) \cup \text{HP}(\Lambda)). \end{aligned}$$

These can be seen in Figures 3. Note that  $D_1$  intersects  $D_0$  on a narrow strip of width 1 on the left of Figure 3b and that  $D_2$  intersects  $D_1$  on an other narrow strip of width 1.

**Definition 6.1.** Let  $n \in \mathbb{N}$ ,  $n \leq N$ . We define

$$\begin{aligned} \tilde{\mathfrak{W}}_n^\pm(\varepsilon; t) &:= \sum_{k=0}^{n-1} \varepsilon^k \tilde{W}_k^\pm(t), \\ \tilde{\mathcal{Z}}_n(\varepsilon; t) &:= \sum_{k=1}^n \varepsilon^k \mathcal{Z}_k(\sigma). \end{aligned}$$

The main result of this section is the following lemma.

**Lemma 6.2.** *There exist  $\Lambda > 1$  and  $\varepsilon_0 > 0$  such that for every  $n \in \mathbb{N}$ ,  $5 \leq n \leq N$  and every  $\varepsilon < \varepsilon_0$  there exists  $C_2 > 0$  such that for all  $t \in D_2$  it holds*

$$\left\| W^-(\varepsilon; t + \frac{\pi\mathbf{i}}{\varepsilon}) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t) \right\|_\infty \leq C_2 \varepsilon^{\frac{n-1}{2}}.$$

<sup>5</sup> This is because  $\Delta_{2\pi\mathbf{i}}[X_k Y_{n-k}](t) = \Delta_{2\pi\mathbf{i}}[X_k](t)Y_{n-k}(t) + X_k(t)\Delta_{2\pi\mathbf{i}}[Y_{n-k}](t)$  implies that  $\Delta_{2\pi\mathbf{i}}[XY](\varepsilon; t) = \Delta_{2\pi\mathbf{i}}[X](\varepsilon; t)Y(\varepsilon; t) + X(\varepsilon; t)\Delta_{2\pi\mathbf{i}}[Y](\varepsilon; t)$ .

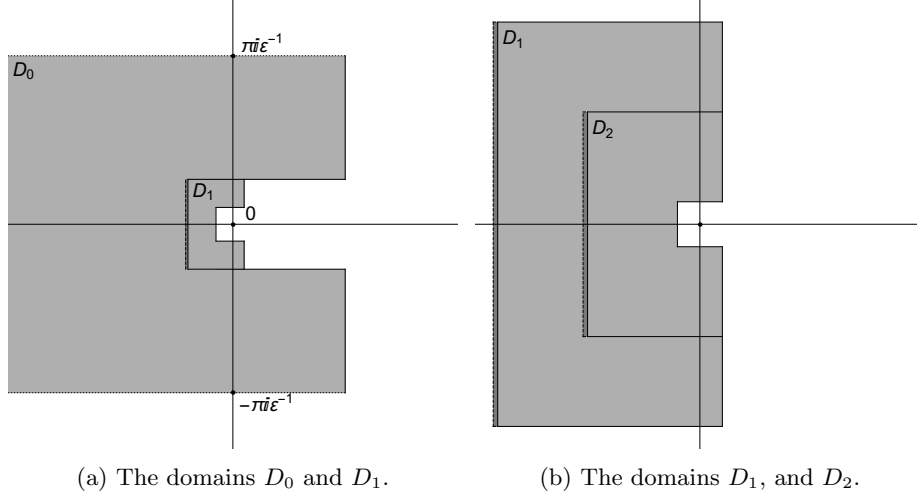


Figure 3: The domains considered.

Moreover for all  $t \in -D_2$  it holds

$$\left\| W^+(\varepsilon; t + \frac{\pi}{\varepsilon} i) - \tilde{\mathfrak{W}}_n^+(\varepsilon; t) \right\|_{\infty} \leq C_2 \varepsilon^{\frac{n-1}{2}}.$$

The rest of this section is devoted to the proof of this lemma.

**Lemma 6.3.** *Let  $A$  be a  $2 \times 2$  matrix. We view  $A$  as a linear map from  $\mathbb{C}^2$  to  $\mathbb{C}^2$ , both equipped with the supremum norm. Then*

$$\|A\|_{\infty} = \max\{|A_{1,1}| + |A_{1,2}|, |A_{2,1}| + |A_{2,2}|\}.$$

*Proof.* Using the definition we get

$$\|A\|_{\infty} = \sup_{\|z\|_{\infty}=1} \|Az\|_{\infty} = \sup_{|z_1|, |z_2| \leq 1} \max\{|A_{1,1}z_1 + A_{1,2}z_2|, |A_{2,1}z_1 + A_{2,2}z_2|\}$$

and from this the result follows.  $\square$

**Lemma 6.4.** *Let  $Q(t) : D_1 \rightarrow \mathbb{C}^2$  and  $c > 0$  such that  $\|Q(t)\|_{\infty} \leq c|t|^{-2}$ . Then there exist  $C_{1,1}, C_{1,2} > 0$  such that*

$$\|F'_{\varepsilon}(\varepsilon Z_1(\sigma) + Q(t))\|_{\infty} \leq 1 + \frac{2}{|t|} + C_{1,1}\varepsilon + \frac{C_{1,2}}{|t|^2}.$$

*Proof.* Let  $s \in \mathbb{C}$ , if  $|s| < 1/2$  it holds

$$\tanh\left(\frac{\pi i}{2} + s\right) = \frac{1}{s} + s\phi(s)$$



with

$$|\phi(s)| \leq 1$$

and

$$\left| \tanh\left(\frac{\pi i}{2} + s\right) \right| \leq \frac{2}{|s|}.$$

So

$$\varepsilon\sigma = \frac{2}{t} + \frac{\varepsilon^2 t}{2} \phi\left(\frac{\varepsilon t}{2}\right),$$

Note that  $\mathcal{F}'_1$  is the identity and

$$\mathcal{F}'_2(x, y) = \begin{pmatrix} -2b_{0,0}y & -2b_{0,0}x + \frac{\varepsilon}{\sqrt{3}} \\ -2b_{0,0}x - \frac{\varepsilon}{\sqrt{3}} & 2b_{0,0}y \end{pmatrix}.$$

This gives

$$\mathcal{F}'_2(\varepsilon\mathcal{Z}_1(\sigma)) = \begin{pmatrix} \varepsilon\sigma & 0 \\ \frac{\sqrt{3}}{2}\varepsilon & -\varepsilon\sigma \end{pmatrix}$$

and  $\|\mathcal{F}'_2(Q(t))\|_\infty \leq \frac{C_1}{|t|^2} + C_2\varepsilon$ .

We have  $F'_\varepsilon(x, y) = \sum_{n \geq 1} \mathcal{F}'_n(\varepsilon; x, y)$ . If  $t \in D_1$  we have that the first component of  $\varepsilon\mathcal{Z}_1(\sigma)$  is a constant times  $\varepsilon$  and the second component is bounded by a constant over  $|t|$ . From this we get

$$\begin{aligned} |\varepsilon\mathcal{Z}_1(\sigma) + Q(t)| &= \frac{1}{|t|} |t\varepsilon\mathcal{Z}_1(\sigma) + tQ(t)| \\ &\leq \frac{1}{2b_{0,0}|t|} \left( \left| \left( 2 + \frac{\varepsilon^2 t^2}{2} \phi\left(\frac{\varepsilon t}{2}\right) \right) \right| + \left( \frac{2b_{0,0}c}{2b_{0,0}|t|} \right) \right) \\ &\leq \frac{1}{2b_{0,0}|t|} \left( \left| 2 + \frac{\varepsilon^2 t^2}{2} \phi\left(\frac{\varepsilon t}{2}\right) \right| + \frac{2b_{0,0}c}{|t|} \right) \\ &\leq \frac{1}{2b_{0,0}|t|} \left( \frac{1}{\sqrt{3}\Lambda} + \frac{2b_{0,0}c}{\Lambda} \right) \\ &\leq \frac{1}{2b_{0,0}|t|} \left( 2 + \frac{1}{2\Lambda^2} \left| \phi\left(\frac{\varepsilon t}{2}\right) \right| + \frac{2b_{0,0}c}{\Lambda} \right) \\ &\leq \frac{1}{2b_{0,0}|t|} \left( \frac{1}{\sqrt{3}\Lambda} + \frac{2b_{0,0}c}{\Lambda} \right) \\ &\leq \frac{1}{2b_{0,0}|t|} \left( 2 + \frac{1}{\Lambda} + \frac{2b_{0,0}c}{\Lambda} \right) = \frac{C_3}{|t|}, \end{aligned}$$

where inequality and absolute value are to be interpreted componentwise. Notice that the constant  $C_3$  is a decreasing function of  $\Lambda$ .

Then we look at  $\mathcal{F}'_n(\varepsilon; x, y)$  for  $n \geq 3$ . Each monomial of  $\mathcal{F}'_n$  is of degree  $n - 1$ . We substitute  $\varepsilon\mathcal{Z}_1(\sigma) + Q(t)$  in  $\sum_{n \geq 3} \mathcal{F}'_n$ , then all the monomials that have a non-zero power of  $\varepsilon$  are in  $O(\varepsilon)$  and all other are in  $O(|t|^{-2})$ .

Collecting everything together we get the result. Notice that the constant  $C_{1,2}$  is a decreasing function of  $\Lambda$ .  $\square$

**Lemma 6.5.** *Let  $Q(t) : D_2 \rightarrow \mathbb{C}^2$  and assume that there exists  $c > 0$  such that  $|Q(t)| \leq c\varepsilon$ . Then there exist  $C_{2,1}, C_{2,2} > 0$  such that*

$$\|F'_\varepsilon(W_0^-(t) + Q(t))\|_\infty \leq 1 + \frac{2}{|t|} + C_{2,1}\varepsilon + \frac{C_{2,2}}{|t|^2}.$$

*Proof.* We take into account that  $|t| > \Lambda$ . Then recall that

$$W_0^-(t) = \begin{pmatrix} 0 \\ -\frac{1}{b_{0,0}t} \end{pmatrix} + r(t)$$

with  $\|r(t)\|_\infty \leq C_r|t|^{-2}$ , which also implies trivially that  $\|W_0^-(t)\|_\infty \leq C_0|t|^{-1}$ , and that

$$F_0(x, y) = \begin{pmatrix} x - 2b_{0,0}xy + b_{0,0}^2x^3 + b_{0,0}^2xy^2 \\ y - b_{0,0}x^2 + b_{0,0}y^2 + b_{0,0}^2x^2y + b_{0,0}^2y^3 \end{pmatrix} + O_4(x, y).$$

From these we get that

$$F'_0(W_0^-(t) + Q(t)) = \begin{pmatrix} 1 + \frac{2}{t} & 0 \\ 0 & 1 - \frac{2}{t} \end{pmatrix} + R(t)$$

with  $\|R(t)\|_\infty \leq C_R(\varepsilon + |t|^{-2})$ , by Lemma 6.3. Moreover  $\forall k \in \mathbb{N}, k \geq 1$  there exists  $C_k$  such that  $\|\varepsilon^k F'_k(W_0^-(t) + Q(t))\|_\infty \leq C_k\varepsilon^k \leq C_k\varepsilon\Lambda^{2-2k}$  and since  $F_\varepsilon$  is analytic around the origin we can sum and get the result. As before the constant  $C_{2,2}$  is a decreasing function of  $\Lambda$ .  $\square$

**Lemma 6.6.** *Let  $\mu : \mathbb{C} \rightarrow \mathbb{R}_+$  with*

$$\mu(t) \leq 1 + \frac{2}{|t|} + c_1\varepsilon + \frac{c_2}{|t|^2}$$

*for some  $c_1, c_2 > 0$ . Then for all  $m \in \mathbb{N}$  with  $m \leq (\Lambda\varepsilon)^{-1} + 2$  it holds*

$$\prod_{k=0}^m \mu(t+k) \leq C \frac{|t|^2}{|t+m|^2}$$

*with*

$$C = \left(1 + \frac{2}{\Lambda} + \frac{c_1 + c_2}{\Lambda^2}\right) \cdot \exp\left(2\pi + \frac{\pi}{\Lambda} \left(c_2 + \left(4 + \frac{c_1}{\Lambda} + \frac{c_2}{\Lambda}\right)^2\right) + c_1 \left(\frac{1}{\Lambda} + \frac{2}{\Lambda^2}\right)\right).$$

*Proof.* For all  $x \in \mathbb{R}$  with  $x \geq 0$ , it holds  $\log(1+x) = x + r(x)$  with  $|r(x)| \leq x^2$ . So we have

$$\begin{aligned} & \log\left(1 + 2\frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1\varepsilon + \frac{c_2}{|t|^2}\right) = \\ & = 2\frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1\varepsilon + \frac{c_2}{|t|^2} + r\left(\frac{1}{|t|} \left(2\frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|} + c_1\varepsilon|t| + \frac{c_2}{|t|}\right)\right) \end{aligned}$$

$$\begin{aligned}
&\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{c_2}{|t|^2} + \frac{1}{|t|^2} \left( 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|} + c_1 \varepsilon |t| + \frac{c_2}{|t|} \right)^2 \\
&\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{1}{|t|^2} \left( c_2 + \left( 4 + \frac{c_1}{\Lambda} + \frac{c_2}{\Lambda} \right)^2 \right) \\
&\leq 2 \frac{|\operatorname{Re}(t)| + |\operatorname{Im}(t)|}{|t|^2} + c_1 \varepsilon + \frac{C_2}{|t|^2}.
\end{aligned}$$

Then by standard integration we get

$$\begin{aligned}
\int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} d\mathfrak{t} &= \log \left( \frac{|t|^2}{|t+m|^2} \right) + 2 \arctan \left( \frac{|\operatorname{Re}(t)| - m}{|\operatorname{Im}(t)|} \right) \\
&\quad - 2 \arctan \left( \frac{|\operatorname{Re}(t)|}{|\operatorname{Im}(t)|} \right) \\
&\leq \log \left( \frac{|t|^2}{|t+m|^2} \right) + 2\pi
\end{aligned}$$

and

$$\int_t^{t+m} \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} = \frac{C_2}{|\operatorname{Im}(t)|} \arctan \left( \frac{|\operatorname{Re}(t)| - m}{|\operatorname{Im}(t)|} \right) - \frac{C_2}{|\operatorname{Im}(t)|} \arctan \left( \frac{|\operatorname{Re}(t)|}{|\operatorname{Im}(t)|} \right) \leq \frac{C_2}{\Lambda} \pi.$$

Also note that

$$c_1 m \varepsilon \leq c_1 \left( \frac{1}{\Lambda} + 2\varepsilon \right).$$

So collecting everything together we get

$$\int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} + \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} + c_1 \varepsilon m \leq \log \left( \frac{|t|^2}{|t+m|^2} \right) + 2\pi + \frac{C_2}{\Lambda} \pi + c_1 \left( \frac{1}{\Lambda} + 2\varepsilon \right),$$

By the above we get

$$\begin{aligned}
\log \left( \prod_{k=1}^m \mu(t+k) \right) &= \sum_{k=0}^{m-1} \log(\mu(t+k)) \\
&\leq \sum_{k=1}^m 2 \frac{|\operatorname{Re}(t+k)| + |\operatorname{Im}(t+k)|}{|t+k|^2} + \frac{C_2}{|t+k|^2} + c_1 \varepsilon \\
&\leq \int_t^{t+m} 2 \frac{|\operatorname{Re}(\mathfrak{t})| + |\operatorname{Im}(\mathfrak{t})|}{|\mathfrak{t}|^2} + \frac{C_2}{|\mathfrak{t}|^2} d\mathfrak{t} + c_1 \varepsilon m \\
&\leq \log \left( \frac{|t|^2}{|t+m|^2} \right) + 2\pi + \frac{C_2}{\Lambda} \pi + c_1 \left( \frac{1}{\Lambda} + 2\varepsilon \right).
\end{aligned}$$

Note that trivially  $\mu(t) \leq 1 + \frac{2}{\Lambda} + \frac{c_1 + c_2}{\Lambda^2}$ . Then exponentiation of the last relation and multiplication by the bound of  $\mu(t)$  gives the result.  $\square$

**Lemma 6.7.** *There exists  $\Lambda > 1$  and  $\varepsilon_0 > 0$  such that for every  $n \in \mathbb{N}$ ,  $5 \leq n \leq N$  and every  $\varepsilon < \varepsilon_0$  there exists  $C_1 > 0$  such that for all  $t \in D_1$  it holds*

$$\left\| W^-(\varepsilon; t + \frac{\pi}{\varepsilon} \mathbf{i}) - \tilde{\mathcal{Z}}_n(\varepsilon; t) \right\|_{\infty} \leq \frac{C_1}{|t|^{n+1}}.$$

*Proof.* Let

$$\begin{aligned}\xi_n(\varepsilon; t) &:= W^-(\varepsilon; t) - \tilde{\mathcal{Z}}_n(\varepsilon; t), \\ R_n(\varepsilon; t) &:= \tilde{\mathcal{Z}}_n(\varepsilon; t) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t - 1)).\end{aligned}$$

It holds  $\tilde{\mathcal{Z}}_n(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t)) = O(\varepsilon^{n+2}\sigma^{n+2})$ . It can be easily checked that  $\tilde{\mathcal{Z}}_1(\varepsilon; t + 1) - F_\varepsilon(\tilde{\mathcal{Z}}_1(\varepsilon; t)) = O(\varepsilon^3\sigma^3)$  and then each order in  $\tilde{\mathcal{Z}}_n$  cancels an order of the difference. So for all  $t \in D_1$  it holds  $R_n(\varepsilon; t) = O(|t|^{-n-2})$ .

Substituting in  $W^-(\varepsilon; t + 1) = F_\varepsilon(W^-(\varepsilon; t))$  we get

$$\xi_n(\varepsilon; t + 1) = \left( \int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t) + \mathfrak{t} \xi_n(\varepsilon; t)) d\mathfrak{t} \right) \xi_n(\varepsilon; t) + R_n(\varepsilon; t + 1),$$

from which we get

$$\xi_n(\varepsilon; t + k + 1) = \left( \int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t + k) + \mathfrak{t} \xi_n(\varepsilon; t + k)) d\mathfrak{t} \right) \xi_n(\varepsilon; t + k) + R_n(\varepsilon; t + k + 1).$$

Let

$$\begin{aligned}\delta_k &:= \|\xi_n(\varepsilon; t + k)\|_\infty, \\ \alpha_k &:= \left\| \int_0^1 F_\varepsilon(\tilde{\mathcal{Z}}_n(\varepsilon; t + k) + \mathfrak{t} \xi_n(\varepsilon; t + k)) d\mathfrak{t} \right\|_\infty, \\ \beta_k &:= \|R_n(\varepsilon; t + k + 1)\|_\infty.\end{aligned}$$

Then we have

$$\delta_{k+1} \leq \alpha_k \delta_k + \beta_k$$

and from this we get that

$$\delta_k \leq \left( \prod_{i=1}^{n-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left( \prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i$$

For all  $t \in D_0 \cap D_1$  it holds

$$\left| W^-(\varepsilon; t) - \tilde{\mathcal{Z}}_n(\varepsilon; t) \right| \leq \frac{C_0}{|t|^{n+1}},$$

so we get  $\delta_0 \leq C_0|t|^{-n-1}$  and  $\beta_k \leq C_\beta|t + k + 1|^{-n-2}$  from Taylor's theorem.

Assume that there exists  $C_1 > \exp(2\pi + 1)(C_0 + C_\beta)$  such that  $\forall j < k$  it holds  $\delta_j \leq C_1|t + j|^{-n-1}$ . Then using Lemma 6.4 we get that

$$\alpha_j \leq 1 + \frac{2}{|t + j|} + C_{1,1}\varepsilon + \frac{C_{1,2}}{|t + j|^2}$$

and

$$\delta_k \leq \left( \prod_{i=1}^{k-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left( \prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i$$

$$\begin{aligned}
&\leq C \frac{|t|^2}{|t+k|^2} \frac{C_0}{|t|^{n+1}} + \sum_{i=0}^{k-1} C \frac{|t+i+1|^2}{|t+k|^2} \frac{C_\beta}{|t+i+1|^{n+2}} \\
&\leq \frac{C_0 C}{|t+k|^2 |t|^{n-1}} + \frac{C_\beta C}{|t+k|^2} \sum_{i=0}^{k-1} \frac{1}{|t+j+1|^n} \\
&\leq \frac{C_0 C}{|t+k|^{n+1}} + \frac{C_\beta C}{|t+k|^2} \frac{1}{|t+k|^{n-1}} \\
&\leq \frac{C(C_0 + C_\beta)}{|t+k|^{n+1}}
\end{aligned}$$

We choose  $\Lambda$  big enough to have<sup>6</sup>  $C(C_0 + C_\beta) < C_1$ . Then we get that the inductive hypothesis holds also for  $m+1$ . This actually proves that the bound is true in  $\text{SQ}_{+1}(\Lambda) \setminus \text{HP}(0)$ . Of course the bound becomes arbitrarily big close to the origin so it will be used only in  $D_1$ . To extend the bound to the whole  $D_1$  we need to apply the same technique for  $\Lambda$  more steps which changes only the constants.  $\square$

*Proof of Lemma 6.2.* Let

$$\begin{aligned}
\xi_n(\varepsilon; t) &:= W^-(\varepsilon; t) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t), \\
R_n(\varepsilon; t) &:= \tilde{\mathfrak{W}}_n^-(\varepsilon; t) - F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t-1)).
\end{aligned}$$

It holds  $\tilde{\mathfrak{W}}_n^-(\varepsilon; t+1) - F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t)) = O(\varepsilon^{n+1}t^{n-1})$ . It can be easily checked that  $\tilde{\mathfrak{W}}_0^-(\varepsilon; t+1) - F_\varepsilon(\tilde{\mathfrak{W}}_0^-(\varepsilon; t)) = O(\varepsilon t^{-1})$  and then each order in  $\tilde{\mathfrak{W}}_n^-$  cancels an order of the difference. So for all  $t \in D_2$  it holds  $R_n(\varepsilon; t) = O(\varepsilon^{n+1}t^{n-1})$ .

Substituting in  $W^-(\varepsilon; t+1) = F_\varepsilon(W^-(\varepsilon; t))$  we get

$$\xi_n(\varepsilon; t+1) = \left( \int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t) + \mathfrak{k} \xi_n(\varepsilon; t)) d\mathfrak{k} \right) \xi_n(\varepsilon; t) + R_n(\varepsilon; t+1),$$

from which we get

$$\xi_n(\varepsilon; t+k+1) = \left( \int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t+k) + \mathfrak{k} \xi_n(\varepsilon; t+k)) d\mathfrak{k} \right) \xi_n(\varepsilon; t+k) + R_n(\varepsilon; t+k+1).$$

Similarly to the above proof we define

$$\begin{aligned}
\delta_k &:= \|\xi_n(\varepsilon; t+k)\|_\infty, \\
\alpha_k &:= \left\| \int_0^1 F_\varepsilon(\tilde{\mathfrak{W}}_n^-(\varepsilon; t+k) + \mathfrak{k} \xi_n(\varepsilon; t+k)) d\mathfrak{k} \right\|_\infty, \\
\beta_k &:= \|R_n(\varepsilon; t+k+1)\|_\infty,
\end{aligned}$$

Then again we have

$$\delta_{k+1} \leq \alpha_k \delta_k + \beta_k$$

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<sup>6</sup> This can always be done since it is equivalent to  $C \leq \exp(2\pi + 1)$ .

and

$$\delta_k \leq \left( \prod_{i=1}^{n-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left( \prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i.$$

From now on we assume that  $t \in D_1 \cap D_2$  and for such  $t$  it holds

$$\left\| W^-(\varepsilon; t) - \tilde{\mathfrak{W}}_n^-(\varepsilon; t) \right\|_\infty \leq \frac{C_1}{|t|^{n+1}} \leq C_1 \varepsilon^{\frac{n+1}{2}},$$

Assume that there exists  $C_2 > \exp(2\pi + 1)$  such that  $\forall j < k$  it holds  $\delta_j \leq C_2 \varepsilon^{\frac{n-1}{2}}$ . Then using Lemma 6.5 we get that

$$\alpha_j \leq 1 + \frac{2}{|t+j|} + C_{1,1} \varepsilon + \frac{C_{1,2}}{|t+j|^2}.$$

and

$$\begin{aligned} \delta_k &\leq \left( \prod_{i=1}^{k-1} \alpha_i \right) \delta_0 + \sum_{i=0}^{k-1} \left( \prod_{j=i+1}^{k-1} \alpha_j \right) \beta_i \\ &\leq C \frac{|t|^2}{|t+k|^2} \frac{C_1}{|t|^{n+1}} + \sum_{i=0}^{k-1} C \frac{|t+j+1|^2}{|t+k|^2} C_\beta \varepsilon^{n+1} |t+j+1|^{n-1} \\ &\leq \frac{C_1 C}{|t+k|^2 |t|^{n-1}} + \frac{C_\beta C \varepsilon^{n+1}}{|t+k|^2} \sum_{i=0}^{k-1} |t+j+1|^{n+1} \\ &\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + \frac{C_\beta C \varepsilon^{n+1}}{|t+k|^2} |t+k|^{n+2} \\ &\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + C_\beta C \varepsilon^{n+1} |t+k|^{n+1} \\ &\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + C_\beta C \varepsilon^{n+1} \left( \frac{\sqrt{2}}{\sqrt{\varepsilon}} \right)^{n+1} \\ &\leq \frac{C_0 C}{|t+k|^2} \varepsilon^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} C_\beta C \varepsilon^{\frac{n+1}{2}} \\ &\leq C \left( \frac{C_0}{|t+k|^2} + 2^{\frac{n+1}{2}} C_\beta \varepsilon \right) \varepsilon^{\frac{n-1}{2}} \\ &\leq C \left( \frac{C_0}{\Lambda^2} + 2^{\frac{n+1}{2}} \frac{C_\beta}{\Lambda^2} \right) \varepsilon^{\frac{n-1}{2}} \end{aligned}$$

Similarly to the previous proof we can choose  $\Lambda$  big enough to get  $\delta_k \leq C_2 \varepsilon^{\frac{n-1}{2}}$ . Then by induction we get the result.  $\square$

Using the inverse map we arrive to a similar result for the stable separatrix.

## 7 Variational equations

There are two variational equations that are very important in this analysis. In this section we will show that the solutions of both can be approximated by the same formal series.

## 7.1 Linear difference equations in a rectangular domain

We consider rectangular symmetric domains around the origin, i.e. there exist  $\alpha, \beta > 0$  such that  $D = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq \alpha, |\operatorname{Im}(z)| \leq \beta\}$ . Let  $\mathcal{O}(D)$  be the space of functions analytic in the interior of  $D$  and continuous at its boundary with the supremum norm over  $D$ .

Let  $g \in \mathcal{O}(D)$ . We will examine the equation

$$X(z+1) - X(z) = g(z). \quad (12)$$

We define the operator

$$\mathcal{S} : X(z) \mapsto X(z+1) - X(z).$$

To solve the equation (12) we need to inverse the operator  $\mathcal{S}$ . We can construct the following two formal solutions

$$\begin{aligned} S^+[g](z) &:= - \sum_{n \geq 0} g(z+n) \\ \text{and} \quad S^-[g](z) &:= \sum_{n \geq 1} g(z-n). \end{aligned}$$

Since  $g$  is defined in a compact set around the origin, the above solutions have no analytic meaning unless  $g$  can be extended beyond its initial domain of definition. Towards this end we have the following lemma.

**Lemma 7.1.** *Let  $h \in \mathcal{O}(D)$ ,  $\chi$  be a Lipschitz continuous function of  $\partial D$  and*

$$J_h = \frac{1}{2\pi} \int_{\partial D} |h(\zeta)| |d\zeta| < \infty.$$

*Then the integral*

$$H(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{h(\zeta)\chi(\zeta)}{\zeta - z} d\zeta$$

*defines two functions  $H_{int}$  and  $H_{ext}$  in the interior and the exterior of  $D$  respectively. Both functions admit continuous extensions onto the closure of their respective domains and*

$$|H_{int,ext}| \leq (J_h + \|h\|_\infty) \|\chi\|_{Lip}.$$

*If  $\operatorname{supp}(\chi) \neq \partial D$  then  $H_{int}$  and  $H_{ext}$  define a single analytic function on  $\mathbb{C} \setminus \operatorname{supp}(\chi)$ .*

*Moreover let  $D$  be contained in a square of side  $R$ . Then*

$$|H_{int,ext}| \leq C \log(R) \|h\|_\infty \|\chi\|_{Lip}$$

*for some  $C > 0$ .*

For a proof see §9 in [Gel99].

We define the function  $\chi^+ : \partial D \rightarrow [0, 1]$  to be Lipschitz continuous. We also ask that  $\chi^+$  has the value 1 on  $\partial D \cap \{z \in \mathbb{C} : \operatorname{Re}(z) < -\alpha/2\}$  and  $\chi^+$  has the value 0 on  $\partial D \cap \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha/2\}$ . We also define  $\chi^-(z) = 1 - \chi^+(z)$ , which implies that  $\|\chi^+\|_{\text{Lip}} = \|\chi^-\|_{\text{Lip}} = L$ . We define

$$h^\pm(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{h(\zeta)\chi^\pm(\zeta)}{\zeta - z} d\zeta.$$

The functions  $h^+$  and  $h^-$  are analytic on  $\mathbb{C} \setminus \operatorname{supp}(\chi^+)$  and  $\mathbb{C} \setminus \operatorname{supp}(\chi^-)$  respectively and  $h^+(z) + h^-(z) = h(z)$  when  $z \in \mathring{D}$  because of the Cauchy integral.

With these we define

$$\mathcal{S}^{-1}[h](z) = S[h](z) := \sum_{n \geq 1} h^-(z - n) - \sum_{n \geq 0} h^+(z + n).$$

This solves equation  $X(z + 1) - X(z) = h(z)$  if both sums are convergent.

In order to generalize this method we need to introduce a weigh function. Let  $\phi_a(z) = e^{az} + e^{-az}$  for some  $a > 0$  and we denote  $\|\phi_a\|_D = \sup_{z \in D} |\phi_a(z)|$ . Then we repeat the above construction with  $h(z) = \phi_a(z)g(z)$ . We define

$$g_a^\pm(z) = \frac{1}{2\pi i \phi_a(z)} \int_{\partial D} \frac{\phi_a(\zeta)h(\zeta)\chi^\pm(\zeta)}{\zeta - z} d\zeta.$$

By definition we have again  $g_a^+(z) + g_a^-(z) = g(z)$  when  $z \in \mathring{D}$ . So we finally define

$$S_a[g](z) := \sum_{n \geq 1} g_a^-(z - n) - \sum_{n \geq 0} g_a^+(z + n).$$

**Lemma 7.2.** *Let  $h \in \mathcal{O}(D)$ ,  $a \geq \frac{\pi}{4\beta}$  and  $r = \max\{2\alpha, 2\beta\}$ . Then  $S_a : \mathcal{O}(D) \rightarrow \mathcal{O}(D)$  and*

$$\|S_a\| \leq C L (1 + a^{-1}) \log(r) \|\phi_a\|_D$$

for some  $C > 0$  and  $S_a[g]$  is a solution of equation (12).

*Proof.* It is trivial to check that formally  $S_a[g]$  is a solution, so we only need to check that the sums converge and get the bound for the norm. For  $z \in \mathring{D}$  and by the previous lemma we have

$$\begin{aligned} |S_a[g](z)| &\leq \left| \sum_{n \geq 1} g_a^-(z - n) \right| + \left| \sum_{n \geq 0} g_a^+(z + n) \right| \\ &\leq C L \log(r) \|\phi_a\|_D \|g\|_\infty \left( \left| \frac{1}{\phi_a(z)} \right| + \sum_{n \geq 1} \left| \frac{1}{\phi_a(z - n)} \right| + \sum_{n \geq 1} \left| \frac{1}{\phi_a(z + n)} \right| \right). \end{aligned}$$

Because  $a \geq \frac{\pi}{4\beta}$ ,  $z$  stays far enough from the roots of  $\phi_a$  so that  $\phi_a(z)^{-1}$  stays bounded by  $1/\sqrt{2}$ . Then both sums can be bounded by some constant times the integral  $\int_0^\infty e^{-as} ds$  and from this we get the result.  $\square$



## 7.2 Approximation of fundamental solutions

The first difference equation is the one that the difference of the separatrices satisfies. We have

$$\begin{aligned}
\delta(\varepsilon; \tau + 1) &= W^+(\varepsilon; \tau + 1) - W^-(\varepsilon; \tau + 1) \\
&= F_\varepsilon(W^+(\varepsilon; \tau)) - F_\varepsilon(W^-(\varepsilon; \tau)) \\
&= \left( \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds \right) (W^+(\varepsilon; \tau) - W^-(\varepsilon; \tau)) \\
&= \left( \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds \right) \delta(\varepsilon; \tau)
\end{aligned}$$

so we write

$$\delta(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \delta(\varepsilon; \tau)$$

with  $A(\varepsilon; \tau) = \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) ds$ . We denote by  $U(\varepsilon; \tau)$  the fundamental solution of this equation, i.e. a  $2 \times 2$  matrix that satisfies

$$U(\varepsilon; \tau + 1) = A(\varepsilon; \tau) \cdot U(\varepsilon; \tau) \quad (13)$$

and  $\det U(\varepsilon; \tau) = 1$ .

For the second variational equation we define  $D(\varepsilon; \tau) = F'_\varepsilon(W^-(\varepsilon; \tau))$  and we denote by  $V(\varepsilon; \tau) = (\Xi(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau))$  a  $2 \times 2$  matrix that satisfies

$$V(\varepsilon; \tau + 1) = D(\varepsilon; \tau) \cdot V(\varepsilon; \tau) \quad (14)$$

and  $\det V(\varepsilon; \tau) = 1$ .

The goal of this section is to prove that we can approximate  $U$  and  $V$  by the same function with errors that are of the same order. To this end we denote by  $R$  the  $2 \times 2$  matrix which satisfies

$$A(\varepsilon; \tau) = D(\varepsilon; \tau) + R(\varepsilon; \tau)$$

and by  $Q$  the  $2 \times 2$  matrix which satisfies

$$U(\varepsilon; \tau) = V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)).$$

Here  $I$  denotes the identity matrix. Then we have

$$\begin{aligned}
U(\varepsilon; \tau + 1) &= V(\varepsilon; \tau + 1)(I + Q(\varepsilon; \tau + 1)) \\
&= D(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau + 1)), \\
A(\varepsilon; \tau)U(\varepsilon; \tau) &= D(\varepsilon; \tau)U(\varepsilon; \tau) + R(\varepsilon; \tau)U(\varepsilon; \tau) \\
&= D(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)) + R(\varepsilon; \tau)V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)).
\end{aligned}$$

From these we get the equation

$$Q(\varepsilon; \tau + 1) - Q(\varepsilon; \tau) = V^{-1}(\varepsilon; \tau) \cdot D^{-1}(\varepsilon; \tau) \cdot R(\varepsilon; \tau) \cdot V(\varepsilon; \tau)(I + Q(\varepsilon; \tau)). \quad (15)$$

**Definition 7.3.** We define the domains

$$\begin{aligned}\mathcal{M}_0 &:= \left\{ \tau \in \mathbb{C} : |\operatorname{Re}(\tau)| \leq 2, |\operatorname{Im}(\tau)| \leq \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} \right\}, \\ \mathcal{M}^\pm &:= \left\{ \tau \in \mathbb{C} : |\operatorname{Re}(\tau)| \leq 2, \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}} \leq \pm \operatorname{Im}(\tau) \leq \frac{\pi}{\varepsilon} - \Lambda \right\}, \\ \mathcal{M} &:= \mathcal{M}_0 \cup \mathcal{M}^+ \cup \mathcal{M}^-.\end{aligned}$$

**Definition 7.4.** Let  $M \in \mathbb{C}^\omega(\mathcal{M})^{2 \times 2}$ . Then we define

$$\|M\|_{\sup} = \max_{i,j \in \{1,2\}} \sup_{t \in \mathcal{M}} |M_{ij}(t)|.$$

**Lemma 7.5.** Let  $n > 8$  and let  $F_\varepsilon$  agree with the normal form up to order  $n$ . Then there exists  $C_V > 0$  such that

$$\|V\|_{\sup} = \frac{C_V}{\varepsilon^4} \left( 1 + O(\varepsilon^{1/2}) \right).$$

*Proof.* By writing  $\Xi(\varepsilon; \tau) = (\xi_1(\varepsilon; \tau), \xi_2(\varepsilon; \tau))$  and  $\dot{W}^-(\varepsilon; \tau) = (\zeta_1(\varepsilon; \tau), \zeta_2(\varepsilon; \tau))$  and using that  $\det(\Xi(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)) = 1$  we get

$$\xi_1(\varepsilon; \tau) = \frac{\zeta_1(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)} \xi_2(\varepsilon; \tau) + \frac{1}{\zeta_2(\varepsilon; \tau)}.$$

Substituting the above relation in equation (14) we get the equation

$$\xi_2(\varepsilon; \tau + 1) = \left( D_{21}(\varepsilon; \tau) \frac{\zeta_1(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)} + D_{22}(\varepsilon; \tau) \right) \xi_2(\varepsilon; \tau) + \frac{D_{21}(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)}.$$

Evidently  $\zeta_2$  satisfies the homogeneous part of the above equation so we define  $\xi_2(\varepsilon; \tau) = C(\varepsilon; \tau) \zeta_2(\varepsilon; \tau)$  and by substitution we get

$$C(\varepsilon; \tau + 1) - C(\varepsilon; \tau) = \frac{D_{21}(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau + 1) \zeta_2(\varepsilon; \tau)} =: K(\varepsilon; \tau).$$

Combining the bounds we got for  $D_0$  and  $D_1$ , for all  $\tau \in \mathcal{M}_0$  we have  $W^-(\varepsilon; \tau) = \tilde{\mathcal{Z}}_n(\varepsilon; \tau) + O(\varepsilon^{\frac{n+1}{2}})$ .

Suppose that  $n > 8$  and that  $\tau \in \mathcal{M}_0$ . We differentiate asymptotic relation for  $W^-(\varepsilon\tau)$  relation to get

$$\begin{aligned}\zeta_2(\varepsilon; \tau) &= \frac{\varepsilon^2}{4b_{0,0}} \operatorname{sech}\left(\frac{\varepsilon\tau}{2}\right) + O(\varepsilon^3 \tanh\left(\frac{\varepsilon\tau}{2}\right)^3) \\ \zeta_1(\varepsilon; \tau) &= O(\varepsilon^3 \tanh\left(\frac{\varepsilon\tau}{2}\right)^3).\end{aligned}$$

The absolute value of  $\operatorname{sech}(z)$  increases as  $\operatorname{Im}(z)$  deviates from 0 and decreases as  $\operatorname{Re}(z)$  deviates from 0. This implies that in  $\mathcal{M}_0$  it is bounded from below by a constant independent of  $\varepsilon$ . We have

$$|\zeta_2(\varepsilon; \tau)^{-1}| = \frac{C'_0}{\varepsilon^2} (1 + O(\varepsilon))$$

for some  $C'_0 > 0$ . In order to bound  $|\zeta_2(\varepsilon; \tau + 1)|$  from below we repeat the above process for  $\tau \in \mathcal{M}_0 + 1$  and we see that the only thing that changes is the constant, i.e.

$$|\zeta_2(\varepsilon; \tau + 1)^{-1}| = \frac{C''_0}{\varepsilon^2} (1 + O(\varepsilon))$$

for some  $C''_0 > 0$ .

To get a bound for  $D_{21}(\varepsilon; \tau)$ , we recall that  $[F'_\varepsilon(x, y)]_{21} = -2a_{0,1}\mu_1\varepsilon - 2b_{0,0}x + 2b_{0,0}^2xy + \dots$  so

$$|D_{21}(\varepsilon; \tau)| = C'''_0\varepsilon + O(\varepsilon^2 \tanh(\frac{\varepsilon\tau}{2}))$$

for some  $C'''_0 > 0$  and since for any  $\tau \in \mathcal{M}_0$  we have  $\varepsilon \tanh(\frac{\varepsilon\tau}{2}) = O(\varepsilon^{1/2})$  we have

$$|K(\varepsilon; \tau)| = \frac{C_0}{\varepsilon^3} (1 + O(\varepsilon^{1/2}))$$

for some  $C_0 > 0$ .

When  $\tau \in \mathcal{M}_1^+$  we need to use  $\tilde{\mathfrak{W}}^-$  to get a bound. From the bound in  $D_2$  we have  $W^-(\varepsilon; \tau) = \tilde{\mathfrak{W}}_n^-(\varepsilon; t) + O(\varepsilon^{\frac{n-1}{2}})$ . We assume again that  $n > 8$ .

Recall that  $\tau = t + \pi i/\varepsilon$ ,  $W_0^-(t) = (0, -(b_{0,0}t)^{-1}) + O(t^{-3})$  and  $W_n^-(t) = O(t^{n-1})$ . Thus  $\dot{W}_0^-(t) = (0, b_{0,0}^{-1}t^{-2}) + O(t^{-4})$ ,  $\dot{W}_1^-(t) = O(t^{-2})$  and  $\dot{W}_n^-(t) = O(t^{n-2})$ .

For  $\tau \in \mathcal{M}_1^+$ ,  $|t|$  is bounded from above by  $\varepsilon^{-\frac{1}{2}}$  and from below by  $\Lambda$ . So in order to bound  $\zeta_2$  from below we need to estimate it for  $\text{Im}(\tau) \approx \frac{\pi}{\varepsilon} - \frac{1}{\sqrt{\varepsilon}}$ . In this region we get

$$\dot{W}_0^-(t) = O(\varepsilon), \dot{W}_1^-(t) = O(\varepsilon) \text{ and } \dot{W}_n^-(t) = O(\varepsilon^{\frac{n}{2}-1}).$$

Using these we get

$$|\zeta_2(\varepsilon; \tau)^{-1}| = \frac{C'_1}{\varepsilon} (1 + O(\varepsilon^{1/2}))$$

for some  $C'_1 > 0$ . As above the same process on  $\mathcal{M}_1^+ + 1$  gives the same bound with a different constant for  $|\zeta_2(\varepsilon; \tau + 1)^{-1}|$ .

Finally, on  $\mathcal{M}_1^+$  we have  $|D_{21}(\varepsilon; \tau)| = C''_1(1 + O(\varepsilon))$  so we get

$$|K(\varepsilon; \tau)| = \frac{C_1}{\varepsilon^2} (1 + O(\varepsilon^{1/2}))$$

for some  $C_1 > 0$ . Due to the real symmetry we get exactly the same bounds on  $\mathcal{M}_1^-$ .

Now that we know that  $K$  is bounded on  $\mathcal{M}$  we can use Lemma 7.2 to get the existence of  $C$ . We set  $a = \varepsilon/2$ , we have  $r = 2\pi\varepsilon^{-1}$ . Then

$$\|S_a\| \leq c'\varepsilon^{-2}$$

and

$$|C(\varepsilon; \tau)| \leq c'' \varepsilon^{-5}.$$

From this we get that

$$\xi_2(\varepsilon; \tau) = C(\varepsilon; \tau) \zeta_2(\varepsilon; \tau) = \frac{c_2}{\varepsilon^3} \left(1 + O(\varepsilon^{1/2})\right)$$

and

$$\xi_1(\varepsilon; \tau) = \frac{1}{\zeta_2(\varepsilon; \tau)} + \frac{\xi_2(\varepsilon; \tau) \zeta_1(\varepsilon; \tau)}{\zeta_2(\varepsilon; \tau)} = \frac{c_1}{\varepsilon^4} \left(1 + O(\varepsilon^{1/2})\right).$$

The maximum of these bounds gives the result.  $\square$

**Lemma 7.6.** *Let  $n \geq 20$  and let  $F_\varepsilon$  agree with the normal form up to order  $n$ . Then there exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon \leq \varepsilon_0$  there exists a constant  $C_Q > 0$  such that  $\|Q\|_\infty \leq C_Q \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2}))$ .*

*Proof.* We know that  $\|D^{-1}\|_{\sup} = 1 + O(\varepsilon^{1/2})$  and since  $\det V = 1$ , we have  $\|V^{-1}\|_{\sup} = C_V \varepsilon^{-4} (1 + O(\varepsilon^{1/2}))$ . We define

$$M = V^{-1} \cdot D^{-1} \cdot R \cdot V.$$

Then equation (15) becomes

$$Q(\tau + 1) - Q(\tau) = M(\tau) + M(\tau) \cdot Q(\tau).$$

From this we get

$$Q(\tau) = S_a[M](\tau) + S_a[M \cdot Q](\tau).$$

We define

$$\mathbf{X} : Q \mapsto S_a[M] + S_a[M \cdot Q].$$

If  $W^+$  and  $W^-$  coincide with the normal form up to order  $n$ , then there exists  $C_n > 0$  such that  $\|R\|_{\sup} = C_n \varepsilon^{\frac{n+1}{2}} (1 + O(\varepsilon^{1/2}))$ . Then there exists  $C_M > 0$  such that  $\|M\|_{\sup} = C_M \varepsilon^{\frac{n-15}{2}} (1 + O(\varepsilon^{1/2}))$ . Recall that  $\|S_a\| \leq c' \varepsilon^{-2}$ , so

$$\|S_a[M]\|_\infty \leq C'_M \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2}))$$

and of course

$$\|S_a[M \cdot Q]\|_\infty \leq C'_M \varepsilon^{\frac{n-19}{2}} (1 + O(\varepsilon^{1/2})) \|Q\|_\infty.$$

Then for  $n \geq 20$  the operator  $\mathbf{X}$  is a contraction on some neighbourhood of the origin  $\mathcal{V}_c = \{x \in \mathbb{C}^\omega(\mathcal{M})^{2 \times 2} : \|x\|_\infty \leq c \|S_a[M]\|_\infty\}$  for big enough  $c$  and small enough  $\varepsilon$ . This means that  $Q$  is the fixed point of  $\mathbf{X}$  and from this the result follows.  $\square$

**Corollary 7.7.**  $U = V + O(\varepsilon^{\frac{n-27}{2}}).$

## 8 Sharper bounds

### 8.1 Upper bound for the splitting

With everything that is known up to this point we can prove that the splitting admits an exponentially small upper bound.

**Lemma 8.1.** *For all  $\tau \in [-2, 2]$  there exists a constant  $C > 0$  such that*

$$|\delta(\varepsilon; \tau)| \leq C\varepsilon^{-2}e^{-\frac{2\pi^2}{\varepsilon}}.$$

Before we prove this lemma we need some results on real analytic periodic functions.

#### Real analytic periodic functions in a rectangular domain

**Lemma 8.2.** *Let  $D = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq \alpha, |\operatorname{Im}(z)| \leq \beta\}$  for some  $\alpha, \beta \geq 1$  and let  $g$  be a real analytic function in  $D$  and continuous on  $\partial D$  such that  $g(\tau + 1) = g(\tau)$  when both  $\tau$  and  $\tau + 1$  are in  $D$ . Moreover, we assume that there exists  $\tau_h \in [-\alpha, \alpha]$  such that  $g(\tau_h) = 0$ . We write  $g$  as a Fourier series:*

$$g(\tau) = g_0 + \sum_{n \geq 1} g_n e^{-2\pi n i \tau} + \sum_{n \geq 1} \overline{g_n} e^{2\pi n i \tau},$$

for some  $g_n \in \mathbb{C}$ . Then it is true that

$$|g_n| \leq \|g\|_{\infty} e^{-2\pi \beta n}$$

for all  $n \in \mathbb{N}$  and

$$|g_0| \leq 4\|g\|_{\infty} e^{-2\pi \beta}.$$

*Proof.* By setting  $\tau = i\beta$  we get

$$g(i\beta) = g_0 + \sum_{n \geq 1} g_n e^{2\pi \beta n} + \sum_{n \geq 1} \overline{g_n} e^{-2\pi \beta n}$$

and this implies that

$$|g_n| \leq \|g\|_{\infty} e^{-2\pi \beta n}$$

for all  $n \in \mathbb{N}_0$ .

From the equation  $g(\tau_h) = 0$  we get

$$|g_0| \leq 2 \sum_{n \geq 1} |g_n|.$$

This sum is a geometric progression so

$$|g_0| \leq 2\|g\|_{\infty} e^{-2\pi \beta} \frac{1}{1 - e^{-2\pi \beta}} \leq 4\|g\|_{\infty} e^{-2\pi \beta}. \quad \square$$

**Corollary 8.3.** *Let  $g$  and  $D$  be as described above. Then for all  $\tau \in [-\alpha, \alpha]$  it is true that*

$$|g(\tau)| \leq 8\|g\|_{\infty} e^{-2\pi \beta}.$$

### Bound for $\delta$

Let  $U(\varepsilon; \tau) = (\Psi(\varepsilon; \tau), \Phi(\varepsilon; \tau))$ . Then there exist two functions  $\Theta(\varepsilon; \tau)$  and  $q(\varepsilon; \tau)$  such that

$$\delta(\varepsilon; \tau) = \Theta(\varepsilon; \tau) \Psi(\varepsilon; \tau) + q(\varepsilon; \tau) \Phi(\varepsilon; \tau).$$

Then evidently we have

$$\Theta(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau))$$

and

$$\begin{aligned} \Theta(\varepsilon; \tau + 1) &= \omega(\delta(\varepsilon; \tau + 1), \Phi(\varepsilon; \tau + 1)) \\ &= \omega(A(\varepsilon; \tau) \delta(\varepsilon; \tau), A(\varepsilon; \tau) \Phi(\varepsilon; \tau)) \\ &= \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau)) \\ &= \Theta(\varepsilon; \tau). \end{aligned}$$

Similarly we get that  $q(\varepsilon; \tau + 1) = q(\varepsilon; \tau)$ .

The map is area-preserving, so there has to be a homoclinic point  $W^-(\varepsilon; \tau_h)$  such that  $\delta(\varepsilon; \tau_h) = 0$ . Because  $\Psi$  and  $\Phi$  are linearly independent this implies that  $\Theta(\varepsilon; \tau_h) = q(\varepsilon; \tau_h) = 0$ .

Both  $\Theta$  and  $q$  are defined in a rectangular domain with  $\alpha = 2$  and  $\beta = \frac{\pi}{\varepsilon} - \Lambda$ . Now we apply Corollary 8.3 and we get that there exists a constant  $C > 0$  such that for all  $\tau \in [-2, 2]$  it holds that

$$|\Theta(\varepsilon; \tau)|, |q(\varepsilon; \tau)| \leq C e^{-\frac{2\pi^2}{\varepsilon}}.$$

To prove Lemma 8.1 we just combine those bounds with the bounds of  $\Psi$  and  $\Phi$ .

## 8.2 Variational equations revisited

In order to prove Lemma 7.6 we used the fact that the stable and unstable solutions can be approximated by the same formal series. This gives an error that is polynomially small with  $\varepsilon$ . However, we saw in the previous section that the splitting is actually exponentially small. We can now use this result to get a sharper bound on the difference of the two fundamental solutions.

**Lemma 8.4.** *Let  $\mathcal{D} = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 2, |\operatorname{Im}(z)| \leq \frac{1}{2}\}$ . Then there exists  $C > 0$  such that on  $\mathcal{D}$  it is true that*

$$\|U - V\|_{\sup} \leq C \varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

*Proof.* The proof is essentially the same as the proof of Lemma 7.6. Here we restate the main points.

By definition we have

$$\begin{aligned} A(\varepsilon; \tau) &= \int_0^1 F'_\varepsilon(s W^+(\varepsilon; \tau) + (1-s) W^-(\varepsilon; \tau)) \, ds \\ &= \int_0^1 F'_\varepsilon(W^-(\varepsilon; \tau) + s \delta(\varepsilon; \tau)) \, ds. \end{aligned}$$

Then

$$\begin{aligned} R(\varepsilon; \tau) &= A(\varepsilon; \tau) - D(\varepsilon; \tau) \\ &= \int_0^1 \left( F'_\varepsilon(W^-(\varepsilon; \tau) + s \delta(\varepsilon; \tau)) - F'_\varepsilon(W^-(\varepsilon; \tau)) \right) \, ds \end{aligned}$$

and by using Taylor's theorem and the bound for  $\delta$  we get that there exists  $C > 0$  such that for all  $\tau \in [-2, 2]$  it holds that

$$|R(\varepsilon; \tau)| \leq C \varepsilon^{-2} e^{-\frac{2\pi^2}{\varepsilon}}.$$

This bound can extend to  $\mathcal{D}$  by increasing the constant.

We have

$$M = V^{-1} \cdot D^{-1} \cdot R \cdot V$$

and by assuming that  $M$  is a function defined on  $\mathcal{D}$  we get

$$\begin{aligned} \|V\|_{\sup}, \|V^{-1}\|_{\sup} &\leq C \varepsilon^{-4} (1 + O(\varepsilon^{1/2})) \\ \|D^{-1}\|_{\sup} &\leq 1 + O(\varepsilon^{1/2}) \\ \|S_a\|_{\sup} &\leq Ch \varepsilon^{-2}. \end{aligned}$$

Recall that we set  $a = \varepsilon/2$ . Then

$$\|S_a[M]\|_{\sup} \leq C \varepsilon^{-12} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

Now by the same contraction mapping argument we get

$$\|Q\|_{\sup} \leq C \varepsilon^{-12} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})),$$

which implies

$$\|U - V\|_{\sup} = \|V \cdot Q\|_{\sup} \leq C \varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})). \quad \square$$

## 9 Asymptotic expansion of the separatrix splitting

We have defined

$$\Theta(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau)),$$

we also define

$$\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)).$$

Note that unlike  $\Theta$ ,  $\Theta^-$  is not periodic, as  $\delta$  and  $\dot{W}^-$  do not satisfy the same equation.

We write  $\Theta$  as a Fourier series

$$\Theta(\varepsilon; \tau) = c_0 + \sum_{n \geq 1} c_n(\varepsilon) e^{-2\pi n i \tau} + \sum_{n \geq 1} \overline{c_n(\varepsilon)} e^{2\pi n i \tau}$$

and from this we get

$$\Theta(\varepsilon; t + \frac{\pi}{\varepsilon} i) = c_0 + \sum_{n \geq 1} c_n(\varepsilon) e^{\frac{2\pi^2 n}{\varepsilon}} e^{-2\pi n i t} + \sum_{n \geq 1} \overline{c_n(\varepsilon)} e^{-\frac{2\pi^2 n}{\varepsilon}} e^{2\pi n i t}.$$

## 9.1 Asymptotic series for $\Theta$

We define

$$L_1(\nu) = \{t \in \mathbb{C} : \text{Im}(t) = -\nu \text{ and } |\text{Re}(t)| \leq \frac{1}{2}\}$$

and we fix  $\nu = -(M+2)(2\pi)^{-1} \log(\varepsilon)$ . This implies that  $e^{2\pi i t} = O(\varepsilon^{-M-2})$ .

**Lemma 9.1.** *There exists a series formal in  $\varepsilon$*

$$\tilde{\Theta}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \zeta_n(t)$$

such that for all  $t \in L_1(\nu)$  it holds that <sup>7</sup>

$$\Theta(\varepsilon; t + \frac{\pi}{\varepsilon} i) = \tilde{\Theta}_N(\varepsilon; t) + O(\varepsilon^{2M+3}).$$

By  $\tilde{\Theta}_N$  we denote the partial sum

$$\tilde{\Theta}_N(\varepsilon; t) = \sum_{n \geq 0}^N \varepsilon^n \zeta_n(t).$$

*Proof.* We define the formal series

$$\tilde{\delta}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \delta_n(t)$$

with  $\delta_n(t) = W_n^+(t) - W_n^-(t)$  and we denote by  $\tilde{\delta}_N$  its truncation to order  $N$ .

We set

$$\begin{aligned} \tilde{\Theta}(\varepsilon; t) &= \omega\left(\tilde{\delta}(\varepsilon; t), \dot{\mathfrak{W}}^-(\varepsilon; t)\right) \\ &= \sum_{n \geq 0} \varepsilon^n \sum_{m=0}^n \omega\left(\delta_m(t), \dot{W}_{n-m}^-(t)\right) \\ &= \sum_{n \geq 0} \varepsilon^n \zeta_n(t). \end{aligned}$$

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<sup>7</sup> Recall that  $N = 6M + 39$ .



Corollary 7.7 implies that for  $t \in L_1(\nu)$  it holds

$$\Phi(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \dot{W}^-(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) + O(\varepsilon^{3M+6}).$$

By lemma 6.2 for  $t \in L_1(\nu)$  we also know that

$$W^\pm(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \sum_{n=0}^N W_n^\pm(t) + O(\varepsilon^{3M+19})$$

which implies that

$$\delta(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \tilde{\delta}_N(\varepsilon; t) + O(\varepsilon^{3M+19}).$$

Recall that  $\delta_n(t) = O(t^{n+2}e^{-2\pi\mathbf{i}t})$ . Then for  $t \in L(\nu)$  we have  $\delta(\varepsilon; t) = O(\varepsilon^{-M-3})$  and these imply that

$$\Theta^-(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) - \Theta(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) = \omega(\delta(\varepsilon; t), W^-(\varepsilon; t) - \Phi(\varepsilon; t)) = O(\varepsilon^{2M+3}).$$

Using the above bounds we get

$$\begin{aligned} \Theta(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}) &= \omega(\delta(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i}), \Phi(\varepsilon; t + \frac{\pi}{\varepsilon}\mathbf{i})) = \\ &= \omega\left(\tilde{\delta}_N(\varepsilon; t)(\varepsilon; t) + O(\varepsilon^{3M+19}), \dot{\mathfrak{W}}_N^-(\varepsilon; t) + O(\varepsilon^{3M+6})\right) \\ &= \omega\left(\tilde{\delta}_N(\varepsilon; t)(\varepsilon; t), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + \omega\left(\tilde{\delta}_N(\varepsilon; t)(\varepsilon; t), O(\varepsilon^{3M+6})\right) \\ &\quad + \omega\left(O(\varepsilon^{3M+19}), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + \omega\left(O(\varepsilon^{3M+19}), O(\varepsilon^{3M+6})\right) \\ &= \omega\left(\tilde{\delta}_N(\varepsilon; t)(\varepsilon; t), \dot{\mathfrak{W}}_N^-(\varepsilon; t)\right) + O(\varepsilon^{3M+6}) \\ &= \sum_{n=0}^N \varepsilon^n \sum_{m=0}^n \omega\left(\delta_m(t), \dot{W}_{n-m}^-(t)\right) + O(\varepsilon^{2M+3}). \end{aligned} \quad \square$$

By the definition of  $\tilde{\Theta}$  we get

$$\begin{aligned} \tilde{\Theta}(\varepsilon; t + 1) &= \omega\left(\tilde{\delta}(\varepsilon; t + 1), \dot{\mathfrak{W}}^-(\varepsilon; t + 1)\right) \\ &= \omega\left(\tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^+(\varepsilon; t)) - \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)), \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \dot{\mathfrak{W}}^-(\varepsilon; t)\right) \\ &= \omega\left(\tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \tilde{\delta}(\varepsilon; t), \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \dot{\mathfrak{W}}^-(\varepsilon; t)\right) \\ &\quad + \omega\left(\tilde{\mathfrak{V}}(\varepsilon; t), \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \dot{\mathfrak{W}}^-(\varepsilon; t)\right) \\ &= \tilde{\Theta}(\varepsilon; t) + \omega\left(\tilde{\mathfrak{V}}(\varepsilon; t), \dot{\mathfrak{W}}^-(\varepsilon; t + 1)\right) \end{aligned}$$

with

$$\tilde{\mathfrak{V}}(\varepsilon; t) = \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t) + \tilde{\delta}(\varepsilon; t)) - \tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) - \tilde{F}'_\varepsilon(\tilde{\mathfrak{W}}^-(\varepsilon; t)) \cdot \tilde{\delta}(\varepsilon; t).$$

**Lemma 9.2.**  $\tilde{\mathfrak{V}}$  can be written as

$$\tilde{\mathfrak{V}}(\varepsilon; t) = \sum_{n \geq 0} \varepsilon^n \mathcal{V}_n(t)$$

with  $\mathcal{V}_n(t) = O(t^{n+4}e^{-4\pi\mathbf{i}t})$ .

*Proof.* In order to show this we need to introduce some new notation. Let  $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be analytic in a neighbourhood of the origin, we write its Taylor series as

$$H(W + v) = \sum_{n \geq 0} \frac{1}{n!} H^{(n)}(W; \underbrace{v, \dots, v}_{n\text{-times}}),$$

where  $H^{(n)}$  has to be viewed as a symmetric tensor. Notice that the tensor is not linear with respect to its first argument.

Using this notation we write  $H'(W) \cdot u$  as  $H^{(1)}(W; u)$  and we have

$$H^{(1)}(W + v; u) = \sum_{n \geq 0} \frac{1}{n!} H^{(n+1)}(W; \underbrace{v, \dots, v}_{n\text{-times}}, u).$$

In general, it holds

$$H^{(n)}(W + u, \underbrace{v, \dots, v}_{n\text{-times}}) = H^{(n)}(W, \underbrace{v, \dots, v}_{n\text{-times}}) + H^{(n+1)}(W, \underbrace{v, \dots, v}_{n\text{-times}}, u) + O(v^n u^2).$$

We also need a slight generalization of the multi-index notation. We define the set

$$\mathcal{P}(n, m) := \left\{ (k_1, \dots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i = n \right\},$$

which is the set of all  $m$ -tuples of positive integers whose sum is  $n$ . Then we define the set

$$\hat{\mathcal{P}}(n, m) := \left\{ (k_1, \dots, \hat{k}_j, \dots, k_m) \in \mathbb{N}^m : \sum_{i=1}^m k_i = n \right\},$$

which is the same as the previous one with the exception that there is a distinguished integer that we mark out. Finally we define the set

$$\hat{\mathcal{P}}_o(n, m) := \left\{ (k_1, \dots, k_{m-1}, \hat{k}_m) \in \mathbb{N}^m : \sum_{i=1}^{m+1} k_i = n \right\},$$

which is a subset of the above, since the distinguished integer is always at the last place.

Using these we define

$$\begin{aligned} \overline{\mathcal{W}}_{(k_1, \dots, k_m)} &= (\dot{W}_{k_1}^-, \dots, \dot{W}_{k_m}^-), \\ \overline{\mathcal{W}}_{(k_1, \dots, \hat{k}_j, \dots, k_m)} &= (\dot{W}_{k_1}^-, \dots, \delta_{k_j}, \dots, \dot{W}_{k_m}^-), \\ \overline{\mathcal{W}}_{(k_1, \dots, k_{m-1}, \hat{k}_m)} &= (\dot{W}_{k_1}^-, \dots, \dot{W}_{k_{m-1}}^-, \delta_{k_m}). \end{aligned}$$

With this notation and having in mind that for any bounded linear map  $A$ , it holds  $A(\delta_i(t), \delta_j(t)) = O(t^k e^{-4\pi i t})$  with  $k = i + j + 4$ , we expand in Taylor series keeping only terms that are independent or linear in any  $\delta_i$  and we get

$$\tilde{F}_\varepsilon(\tilde{\mathfrak{W}}^- + \tilde{\delta}) = \sum_{n \geq 0} \varepsilon^n \left( \tilde{F}_n(W_0^- + \delta_0) + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m, k)} \frac{1}{k!} \tilde{F}_m^{(k)}(W_0^- + \delta_0; \overline{\mathcal{W}}_p) \right)$$

$$+ \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \hat{\mathcal{P}}(n-m,k)} \frac{1}{k!} \tilde{F}_m^{(k)}(W_0^- + \delta_0; \overline{\mathcal{W}}_p) + O(t^{n+4}e^{-4\pi i t}) \Bigg),$$

$$\begin{aligned} \tilde{F}'_\varepsilon(\tilde{\mathcal{W}}^-(\varepsilon; t)) &= \sum_{n \geq 0} \varepsilon^n \left( \tilde{F}_n(W_0^-) \right. \\ &\quad \left. + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m,k)} \frac{1}{k!} \tilde{F}_m^{(k)}(W_0^-; \overline{\mathcal{W}}_p) + O(t^{n+4}e^{-4\pi i t}) \right), \end{aligned}$$

and

$$\begin{aligned} \tilde{F}'_\varepsilon(\tilde{\mathcal{W}}^-(\varepsilon; t)) \cdot \delta(\varepsilon; t) &= \sum_{n \geq 0} \varepsilon^n \left( \sum_{m=0}^n \tilde{F}_m^{(1)}(W_0^-; \delta_{n-m}) \right. \\ &\quad + \sum_{m=0}^{n-1} \sum_{k=1}^{n-m} \sum_{p \in \mathcal{P}(n-m,k)} \frac{1}{k!} \tilde{F}_m^{(k+1)}(W_0^-; \overline{\mathcal{W}}_p, \delta_0) \\ &\quad + \sum_{m=0}^{n-1} \sum_{k=2}^{n-m} \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(W_0^-; \overline{\mathcal{W}}_p) \\ &\quad \left. + O(t^{n+4}e^{-4\pi i t}) \right). \end{aligned}$$

We fix  $n, m, k$  and  $p \in \mathcal{P}(n-m, k)$ , then we see that

$$\frac{1}{k!} \tilde{F}_m^{(k)}(W_0^- + \delta_0; \overline{\mathcal{W}}_p) - \frac{1}{k!} \tilde{F}_m^{(k)}(W_0^-; \overline{\mathcal{W}}_p) - \frac{1}{k!} \tilde{F}_m^{(k+1)}(W_0^-; \overline{\mathcal{W}}_p, \delta_0)$$

is of order  $\delta_0^2$ .

Then we fix  $n, m$ , set  $k = 1$  and we see that

$$\tilde{F}_m^{(1)}(W_0^- + \delta_0; \delta_{n-m}) - \tilde{F}_m^{(1)}(W_0^-; \delta_{n-m})$$

is of order  $\delta_0 \delta_{n-m}$ .

Finally we fix  $n, m, k > 1$  and we see that

$$\begin{aligned} &\sum_{p \in \hat{\mathcal{P}}(n-m,k)} \frac{1}{k!} \tilde{F}_m^{(k)}(W_0^- + \delta_0; \overline{\mathcal{W}}_p) - \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(W_0^-; \overline{\mathcal{W}}_p) \\ &= \sum_{p \in \hat{\mathcal{P}}_o(n-m,k)} \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(W_0^- + \delta_0; \overline{\mathcal{W}}_p) - \frac{1}{(k-1)!} \tilde{F}_m^{(k)}(W_0^-; \overline{\mathcal{W}}_p). \end{aligned}$$

This implies each term of the sum is of order  $\delta_0 \delta_j$  for some  $j \in \{1, \dots, n-m\}$ .

The above show that for all  $n$ ,  $\mathcal{V}_n(t) = O(t^{n+4}e^{-4\pi i t})$ .  $\square$

## 9.2 The first Fourier coefficient of $\Theta$

We define

$$\theta(\varepsilon) := \int_{L_1(\nu)} e^{2\pi i t} \Theta(\varepsilon; t + \frac{\pi}{\varepsilon} i) dt = c_1(\varepsilon) e^{\frac{2\pi^2}{\varepsilon}}$$

with  $c_1(\varepsilon)$  denoting the first Fourier coefficient in the expansion of  $\Theta$ . with  $\nu > 0$  and  $L_1(\nu) = \{t \in \mathbb{C} : \text{Im}(t) = -\nu \text{ and } |\text{Re}(t)| \leq \frac{1}{2}\}$ . Notice that since the function  $\Theta$  is periodic,  $\theta(\varepsilon)$  does not depend on  $\nu$ .

Lemma 9.1 implies that

$$\begin{aligned} \theta(\varepsilon) &= \int_{L_1(\nu)} e^{2\pi i t} \left( \tilde{\Theta}_N(\varepsilon; t) + O(\varepsilon^{2M+3}) \right) dt \\ &= \sum_{n=0}^N \varepsilon^n \left( \int_{L_1(\nu)} e^{2\pi i t} \zeta_n(t) dt \right) + O(\varepsilon^{M+1}). \end{aligned}$$

**Lemma 9.3.** *There exist constants  $\theta_i \in \mathbb{C}$  such that*

$$\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1}).$$

To prove this lemma we need to show that  $\Theta$  can be approximated using the formal solution.

*Proof.* Recall that it holds

$$\theta(\varepsilon) = \sum_{n=0}^N \varepsilon^n \left( \int_{L_1(\nu)} e^{2\pi i t} \zeta_n(t) dt \right) + O(\varepsilon^{M+1}). \quad (16)$$

We define

$$L_1^-(\mu) := \bigcup_{\kappa \geq \mu} L_1(\kappa)$$

and

$$L^-(\mu) := \bigcup_{\kappa \geq \mu} \left( L_1(\kappa) \cup (L_1^-(\kappa) + 1) \right).$$

Since  $\Theta$  is periodic in  $t$ , it holds

$$\zeta_n(t+1) = \zeta_n(t) + \sum_{m=0}^n \omega(\mathcal{V}_m(t), \dot{W}_{n-m}^-(t+1))$$

and this implies that

$$e^{2\pi i(t+1)} \zeta_n(t+1) = e^{2\pi i t} \zeta_n(t) + r_n(t)$$

with

$$r(t) = \sum_{m=0}^n \omega(e^{2\pi i t} \mathcal{V}_m(t), \dot{W}_{n-m}^-(t+1))$$

and  $r_n(t) = O(t^{n+2}e^{-2\pi it})$ . All of the above functions are analytic in  $L^-(\nu)$ .

For all  $t \in L_1^-(\nu)$  we define

$$\rho_n(t) = \int_t^{t+1} e^{2\pi is} \zeta_n(s) ds.$$

This satisfies the equation

$$\rho_n(t+1) = \rho_n(t) + \int_t^{t+1} r_n(s) ds,$$

which has as solution

$$\rho_n(t) = \theta_n + \int_{-i\infty}^t r_n(s) ds$$

for some constant  $\theta_n$ .

Since we know the bound for  $r_n$  and it holds

$$\int_{-\infty}^{|t|} s^{n+2} e^{-2\pi s} ds \leq C_\nu |t|^{n+2} e^{-2\pi |t|}$$

we get that for all  $t \in L_1^-(\nu)$

$$\rho_n(t) = \theta_n + O(t^{n+2}e^{-2\pi it}).$$

As  $\nu$  was chosen such that  $e^{-2\pi it} = O(\varepsilon^{M+2})$ , then for any  $n \in \mathbb{N}$  it holds  $t^{n+2}e^{-2\pi it} = O(\varepsilon^{M+1})$ . This gives

$$\int_{L_1(\nu)} e^{2\pi it} \zeta_n(t) dt = \theta_n + O(\varepsilon^{M+1}),$$

which we can combine with the equation (16) to get

$$\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1}). \quad \square$$

**Remark.** For the first constant  $\theta_0$  we have

$$\theta_0 = \int_{L_1(\nu)} e^{2\pi it} \omega(\delta_0(t), \dot{W}_0^-(t)) dt,$$

which is approximately the Stokes constant of the resonant map.

### 9.3 The constant term of $\Theta$

**Lemma 9.4.** *There exists  $C > 0$  such that*

$$|c_0| \leq C \varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

*Proof.* Let  $\tau_h$  be such that  $W^+(\varepsilon; \tau_h) = W^-(\varepsilon; \tau_h)$ , then  $W^+(\varepsilon; \tau_h + 1) = W^-(\varepsilon; \tau_h + 1)$ . By definition we have that

$$c_0 = \int_0^1 \omega(\delta(\varepsilon; \tau_h + s), \Phi(\varepsilon; \tau_h + s)) ds.$$

We define

$$\sigma_0 = \int_0^1 \omega(\delta(\varepsilon; \tau_h + s), \dot{W}^-(\varepsilon; \tau_h + s)) ds,$$

so

$$|c_0 - \sigma_0| \leq \int_0^1 |\omega(\delta(\varepsilon; \tau_h + s), \Phi(\varepsilon; \tau_h + s) - \dot{W}^-(\varepsilon; \tau_h + s))| ds.$$

Using lemma 8.1 and the bound for  $\delta$  we get that

$$|c_0 - \sigma_0| \leq C\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2})).$$

Let  $A$  be the signed area enclosed by these two pieces of the separatrices. Using Green's formula to calculate the area we get that

$$A = \frac{1}{2} \int_0^1 \omega(W^+(\varepsilon; \tau_h + s), \dot{W}^+(\varepsilon; \tau_h + s)) - \omega(W^-(\varepsilon; \tau_h + s), \dot{W}^-(\varepsilon; \tau_h + s)) ds.$$

It holds that  $W^+ = W^- + \delta$  so we have

$$\omega(W^+, \dot{W}^+) - \omega(W^-, \dot{W}^-) = \omega(\delta, \dot{W}^-) + \omega(W^-, \dot{\delta}) + \omega(\delta, \dot{\delta})$$

and from these

$$\begin{aligned} \omega(\delta, \dot{W}^-) - \frac{1}{2} (\omega(W^+, \dot{W}^+) - \omega(W^-, \dot{W}^-)) &= \frac{1}{2} (\omega(\delta, \dot{W}^-) - \omega(W^-, \dot{\delta}) - \omega(\delta, \dot{\delta})) \\ &= \frac{1}{2} \left( \frac{d}{dt} \omega(\delta, W^-) - \omega(\delta, \dot{\delta}) \right). \end{aligned}$$

Combining the above we get

$$\sigma_0 - A = \omega(\delta(\varepsilon; \tau_h + s), W^-(\varepsilon; \tau_h + s)) \Big|_{s=0}^1 - \int_0^1 \frac{1}{2} \omega(\delta(\varepsilon; \tau_h + s), \dot{\delta}(\varepsilon; \tau_h + s)) ds.$$

We know that the  $\delta$  and  $\dot{\delta}$  are bounded by  $C\varepsilon^{-2} e^{-\frac{2\pi^2}{\varepsilon}}$  and since the map is area-preserving we get that  $A = 0$ . This leads to the result.  $\square$

## 9.4 Asymptotic series for the homoclinic invariant

Now we have all the ingredients we need to prove the asymptotic for the Lazutkin homoclinic invariant.

**Lemma 9.5.** *There exist real numbers  $\vartheta_n$  such that*

$$\Omega(\varepsilon) = \left( \sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1}) \right) e^{-\frac{2\pi^2}{\varepsilon}}.$$

Moreover  $\vartheta_0 = 4\pi|\theta_0|$ , where  $\theta_0$  is the Stokes constant of the resonant map.

*Proof.* By lemma 8.4 we have

$$\Phi(\varepsilon; \tau) - \dot{W}^-(\varepsilon; \tau) = O(\varepsilon^{-16} e^{-\frac{2\pi^2}{\varepsilon}}).$$

This implies that since

$$\Theta(\varepsilon; \tau) - \Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \Phi(\varepsilon; \tau) - \dot{W}^-(\varepsilon; \tau)),$$

using the improved bound of lemma 8.1 for real  $\tau$  we get

$$\Theta(\varepsilon; \tau) - \Theta^-(\varepsilon; \tau) = O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}})$$

and we know that

$$\Theta(\varepsilon; \tau) = c_0 + \theta(\varepsilon) e^{-\frac{2\pi^2}{\varepsilon}} e^{-2\pi i \tau} + \overline{\theta(\varepsilon)} e^{-\frac{2\pi^2}{\varepsilon}} e^{2\pi i \tau} + O(e^{-\frac{4\pi^2}{\varepsilon}}).$$

Since we know that  $|c_0| \leq C \varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}} (1 + O(\varepsilon^{1/2}))$ , we get

$$\Theta^-(\varepsilon; \tau) = 2|\theta(\varepsilon)| e^{-\frac{2\pi^2}{\varepsilon}} \cos(2\pi\tau - \arg(\theta(\varepsilon))) + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}}).$$

Let  $W^+(\tau_h)$  be a homoclinic point. Then evidently  $\Theta^-(\varepsilon; \tau_h) = 0$  and from the above relation we get that

$$2\pi\tau_h - \arg(\theta(\varepsilon)) = 2\pi k + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}})$$

for some  $k \in \mathbb{Z}$ . This implies that

$$\dot{\Theta}^-(\varepsilon; \tau_h) = 4\pi|\theta(\varepsilon)| e^{-\frac{2\pi^2}{\varepsilon}} + O(\varepsilon^{-18} e^{-\frac{4\pi^2}{\varepsilon}}).$$

Differentiating  $\Theta^-(\varepsilon; \tau) = \omega(\delta(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau))$  we get

$$\dot{\Theta}^-(\varepsilon; \tau) = \omega(\dot{\delta}(\varepsilon; \tau), \dot{W}^-(\varepsilon; \tau)) + \omega(\delta(\varepsilon; \tau), \ddot{W}^-(\varepsilon; \tau))$$

Since  $\delta(\varepsilon; \tau_h) = 0$  we get

$$\begin{aligned} \dot{\Theta}^-(\varepsilon; \tau_h) &= \omega(\dot{\delta}(\varepsilon; \tau_h), \dot{W}^-(\varepsilon; \tau_h)) \\ &= \omega(\dot{W}^+(\varepsilon; \tau_h), \dot{W}^-(\varepsilon; \tau_h)), \end{aligned}$$

which is by definition the homoclinic invariant.

Finally to prove the lemma we use the fact that  $\theta(\varepsilon) = \sum_{n=0}^M \varepsilon^n \theta_n + O(\varepsilon^{M+1})$ . This implies that

$$4\pi|\theta(\varepsilon)| = \sum_{n=0}^M \vartheta_n \varepsilon^n + O(\varepsilon^{M+1})$$

for some real constants  $\vartheta_n$ . □

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